DynamicsOfPolygons.org

Winding numbers and density plots

Winding Numbers and Density Plots

To analyze complex orbits , we will use a tool borrowed from classical analysis. The winding number of an orbit is a measure of the average amount of rotation per period.

Definition: For a given N-gon, and a point p, let S = s_1, s_2,.. be the step sequence (number of corners advanced on each iteration). The winding number of S is defined to be

$$\omega(S) = \frac{1}{N}[\lim_{m\to\infty} \frac{1}{m}\sum_{j=1}^{m} s_j]$$

Since the minimum step size is 1, ω(S) is bounded below by 1/N and for a regular N-gon this is obtained by the canonical 1-step orbit. The remaining canonical orbits give an increasing sequence of winding numbers 1/N, 2/N,… which is bounded above by 1/2 and this limiting value corresponds to the ideal 'horizon orbit' with maximum average step size.

If S is a periodic step sequence with prime period k then $\omega(S) = \frac{1}{N}[\frac{1}{k}\sum_{j=1}^{k} s_j]$

So in this case the winding number is 1/N times the mean number of steps in one period. For example, with N = 7, Dad has step sequence (3) so his winding number is 3/7. Helen's step sequence is (32) so her winding number is 5/14. The winding number of the dense orbit for N = 5, approaches .25.

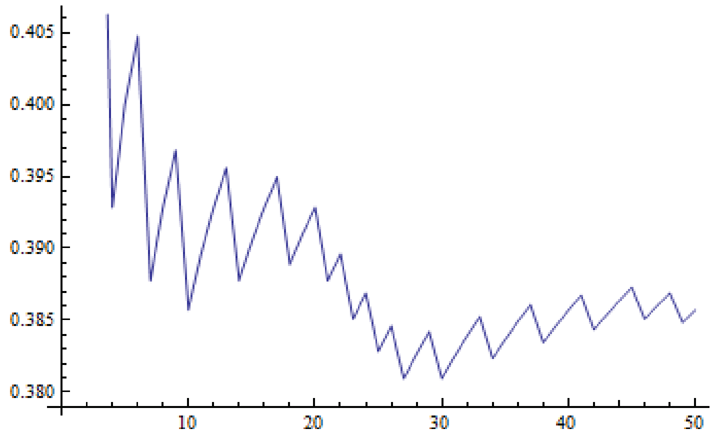
In Mathematica **Vp[p,k]** will generate the first k terms in the step sequence of p.

Example: Using N = 7, the center of Mom[5] is q ≈ {-3.94733919670,-0.900954412853005} (period 433468)

To estimate the winding number of q, define : **Win[S_] := N[Plus@@S/(Length[S]*npoints)];** Here we will use just the first 5000 so **S = Vp[q, 5000]** and Win[S] = .380343

To plot the evolution of ω: **Win[S_,n_] := N[Plus@@Take[S,n]/(n*npoints)];**

Tb=Table[Win[S,i],{i,1,50}]; ListPlot[Tb, PlotJoined->True]
(*sample at intervals of 50 and plot*)



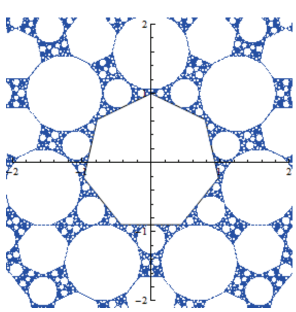
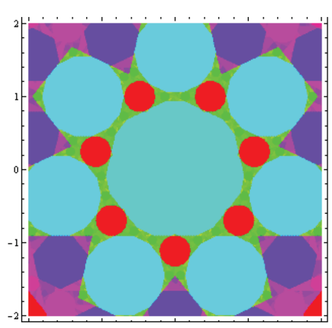
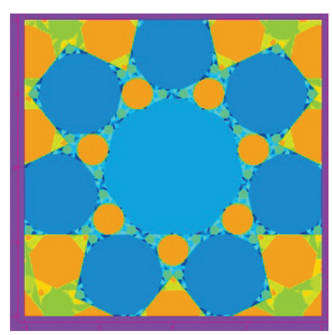
Density Plots

Points with similar dynamics will usually have similar winding numbers so a scan of winding numbers can provide a visual image of the dynamics. These are called density plots and the variations in density can be plotted as colors or heights for a 3-simensional plot.

Example 1: To scan the region around the origin for N = 7. This region is shown on the left below. (The Tangent Map is perfectly happy to track points inside M. The first iteration just reflects them outside and the orbit is normal from there.)

Win[x_,y_]:=N[Plus@@Vp[{x,y}, 40]]; (*unscaled winding numbers - On density plots Mathematics does its own scaling to (0,1) *)
DensityPlot[Win[x,y],{x,-2,2},{y,-2,2}, PlotPoints->200, ColorFunction->Hue] (*about 2 minutes*)

The Density plot is the middle image and the right-hand image is the Photoshop enhancement of this image. The density plot easily picks out the invariant inner star region - because the points in this region have step sequences with 1 and 2 vs. 2 and 3 for the outer region.

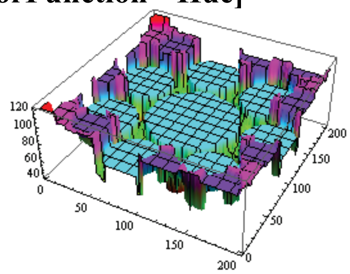




Generating detailed density plots is slow and the results are hard to predict. There is much more flexibility in using tables of values and plotting them using ListDensityPlot or ListPlot3D. The tables are ordinary matrices which can be generated in sections and appended together.

To do the above plot with a table: **Win[x_,y_]:=N[Plus@@Vp[{x,y},50]];**

Tb =Table[Win[x,y],{x,-2,2,.02},{y,-2,2,.02}]; (*about 2 minutes*)
ListDensityPlot[Tb, ColorFunction->Hue] (*about 10 seconds*)
The result is identical to the above, but now we have a table which can be saved or manipulated. The same table can be used to generate a 3D plot:

ListPlot3D[Tb,ColorFunction->Hue]



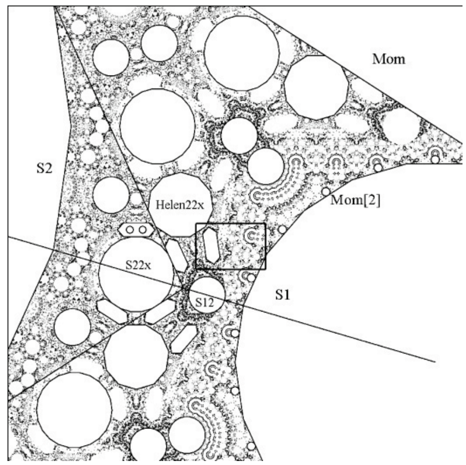
For more detailed plots, it may be necessary to partition the tables to keep the file size manageable. The routine WT shown below generates the table Tb in steps, so WT[4,150] will use the current 'crop' region and the current winding number depth, to partition Tb into 4 sub-tables, each with width 150.

```
WT[steps_,length_]:=Module[{dx=(right-left)/length,dy=(top-bottom)/length},
Tb = Range[steps];
For[j=1,j<=steps,j++,Tb[[j]]=Transpose[Table[Win[x,y],{x,left,right-
dx,dx},{y,bottom+dy*(j-1)*(length/steps),bottom+dy*j*(length/steps)-dy,dy}]];
];Return[Tb]];
```

For highly detailed plots, the Tb file may be too large to generate in one session. The 'extended' version of WT below, saves the intermediate files so that they can be joined at a later time. WT[4,150,"temp"] will create and save files called "tempk" for k = 1 to 4. This allows the user to run long plots overnight or on multiple platforms if necessary. Each file can be plotted separately or joined together for a full plot. Memory usage is a major factor for large tables and this routine tracks the status of MemoryInUse to guard against crashes.

```
WT[steps_, length_, name_] :=
  Module[{dx = (right - left)/length, dy = (top - bottom)/length},
   J = Range[steps]; Unprotect[TB]; Print[MemoryInUse[]];
   For[j = 1, j <= steps, j++,
    TB = Table[
      Win[x, y], {x, left, right - dx, dx}, {y,
       bottom + dy*(j - 1)*(length/steps),
       bottom + dy*j*(length/steps) - dy, dy}];
    Print[j]; Save[name <> ToString[j], TB]; TB =.;
    Print[MemoryInUse[]]]];
```

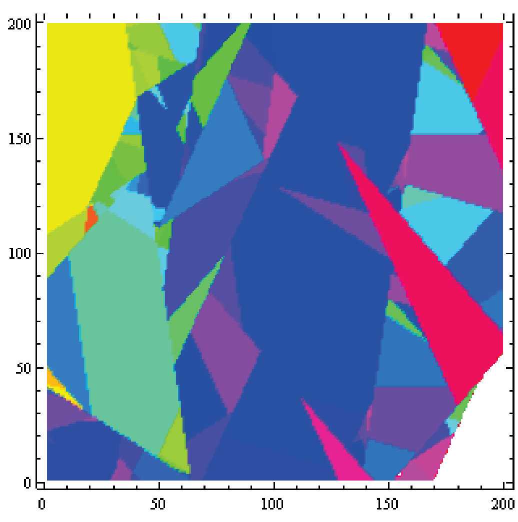
Example 2:In N = 11, the region between S1 and S2 is very complex as shown in the plot below. We will make a rough density plot of the outlined region and compare that plot with a detailed plot generated over a period of a few days.



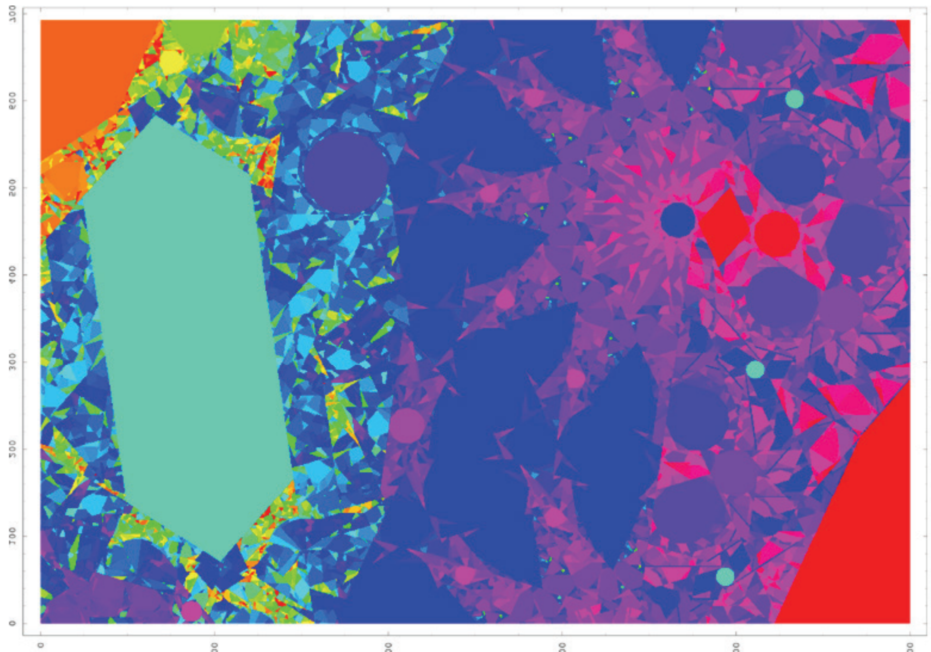
The central point in this region is q = {-0.647039, -0.8230585} and box[q, .015] will yield a box with width and height of .03. We will use a density of 150, so **Win[x_,y_]:=N[Plus@@Vp[{x,y}, 150]];**

Using the 'short' version of WT: **WT[4, 200]** will generate Tb (* about 8 minutes*) ,which consists of matrices Tb[[1]] through Tb[[4]] and each of these will have Dimensions of {50,200}. **Tc = Join[Tb[[1]],Tb[[2]], Tb[[3]], Tb[[4]]]** will stack these matrices one on top of the other to generate a matrix Tc with dimensions{200,200}

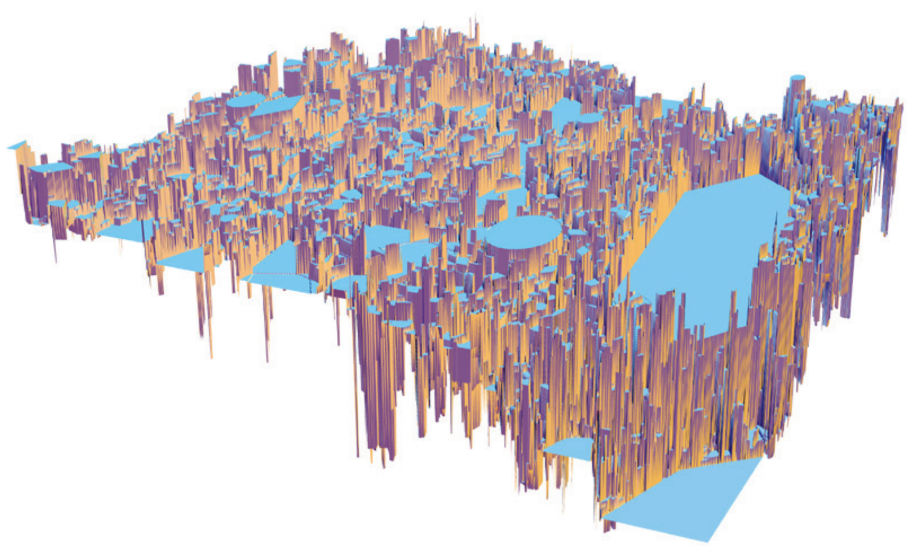
ListDensityPlot[Tc,ColorFunction->Hue]



Compare this with the 1000 by 1000 plot below at depth 3000

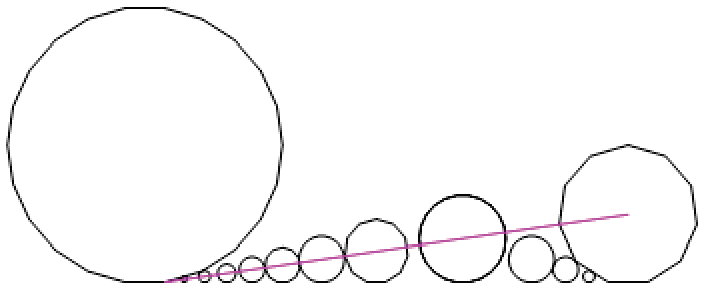


Using the same table from the plot above**: ListPlot3D[Tb]** will generate the 3D version and allow the user to view the region from any perspective.



The view here is left to right compared to the density plot, so it is looking from S2 toward S1. Since S1 is step-1, it has the lowest possible winding number of ω = 1/11 and this means it is virtually invisible in a deep hole at the far side of this plot. The 'skating rink' is on high ground with ω = 41/169. It is not unusual for such regions to be surrounded by 'halos' of energetic points.

Example 3. For all regular polygons, the major line of symmetry is the line joining the origin to GenStar, as shown below for N = 11

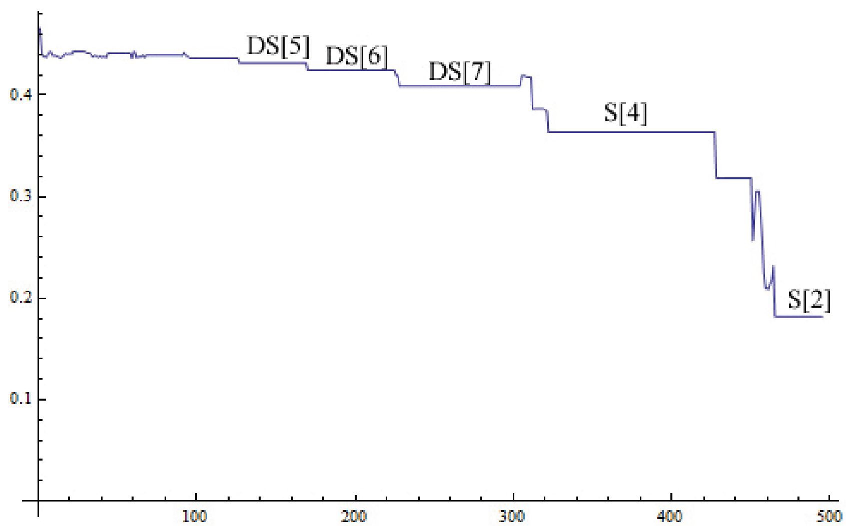


It is an easy matter to scan this line and generate a profile of the dynamics. The slope of this line is called cslope1 (= -h0/GenStar[[1]]). The equation is **fc[x_]:= cslope1*x**.
If **d1 = EuclideanDistance[M[9],GenStar]** then to generate a plot with 500 steps:

Tbx=Table[Win[x,fc[x]],{x,GenStar[[1]],M[9][[1]], d1/500}]; (*about 2 minutes using the depth 500 Win function below*)

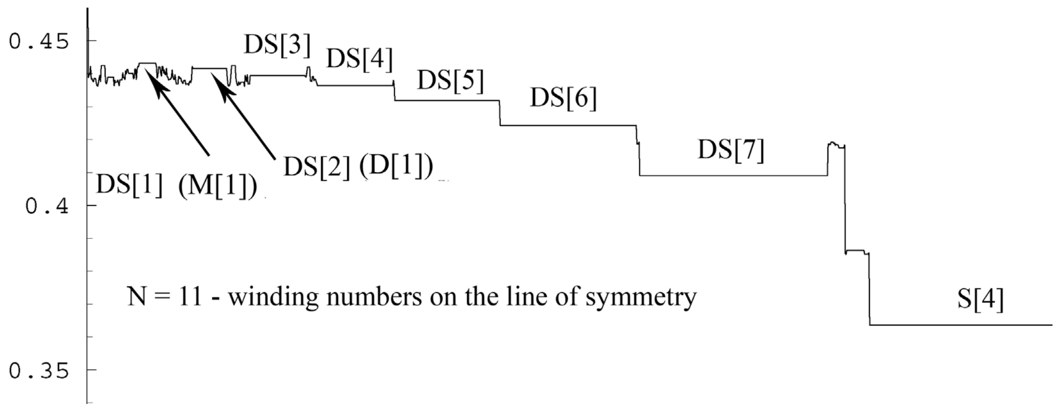
Win[x_,y_]:=N[Plus@@Vp[{x,y}, 500]]; (*we will scale these when we plot them*)

ListPlot[Tbx/5500, AxesOrigin-> {0,0}, PlotJoined->True, PlotStyle -> AbsolutePointSize[0.1], PlotRange->All]



The smallest possible winding number ω = 1/11 does not appear here because S[1] is not on this center line. The smallest value above is S[2] at 2/11. The transitions between these major buds is far from trivial and would show detail on all possible scales. The limiting value of ω for all orbits is .5 but no orbit has this winding number. In the star region shown above, the largest ω is Dad at 5/11 = .454545...

Below is a close up of the GenStar region. The peak to the left of M[1] is M[2] but there is no D[2] so the sequence of generations does not continue. The region to the left of M[1] is poorly understood.



The highest peak is M[1] at 39/88 since her step sequence is (55555554). D[1] with (5555554) is the wider plateau on her right. The small peak to the left of M[1] is M[2] whose step sequence is a mixture of M[1] and D[1] so it is not surprising that her winding number is between these two values. It is not clear whether there are periodic orbits with winding numbers arbitrarily close to D at 5/11 because the family structure breaks down after M[2]. We do not know what step sequences are admissible and this of course means that we do not know what periods are admissible.

Below is a density plot of the transition region between S[4] and DS[7]. This is an important transition because it is defines the boundary of the inner star region. The major 'steps' here can also be observed in the plot above. The plateau is part of the border of DS[3] - who can just be seen at the top of the plot. Likewise S[4] can just be see in the foreground and the rest of the foreground is his border.