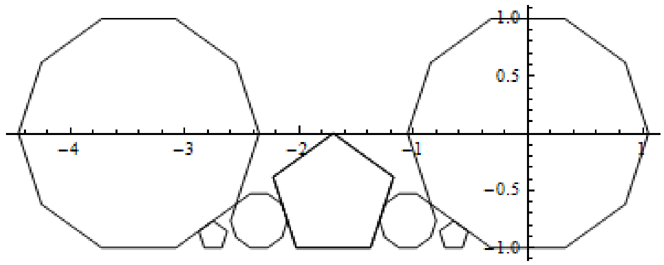
DynamicsOfPolygons.org

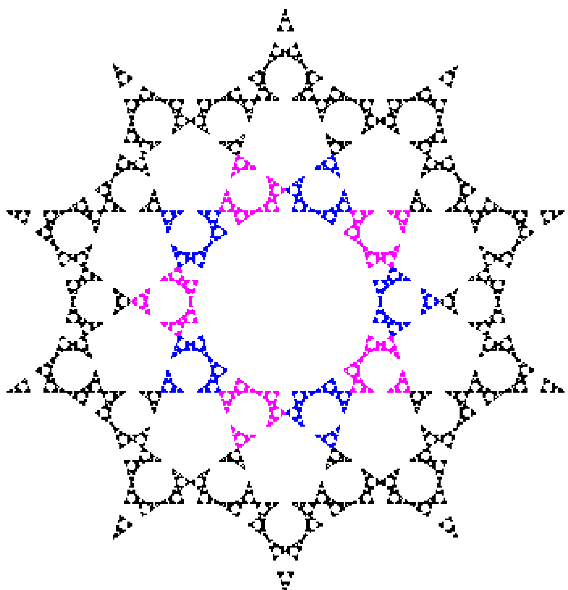
Summary of dynamics of the regular decagon: N = 10

N = 10 and N = 5 share the same complexity and scaling, so their webs are identical, but the dynamics are different because of the change of origin.

Graphics[poly/@FirstFamily, Axes->True]

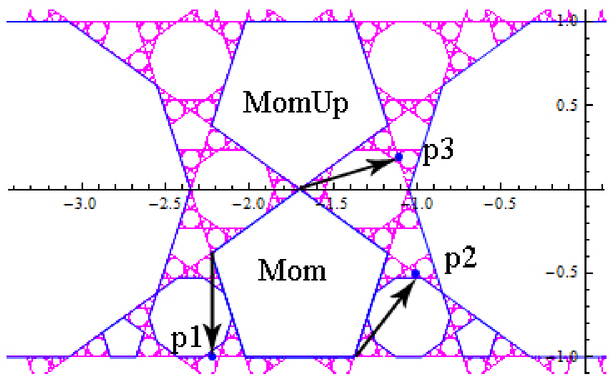


All composite N-gons have decomposition of orbits so they will have multiple invariant regions within the inner star region. Below are the three invariant regions for N = 10



This decomposition has no effect on the fractal dimension – which is the same as N = 5, but now it will require 3 non-periodic orbits to generate this region.

Using M and the matching MUp ,we can find three non-periodic points that will generate the three invariant regions for D. The point p1 is the 'canonical' non periodic point back in N = 5.
p1= {M[[5]][[1]],M[[4]][[2]]} ≈ {-2.227032728823213470388750078 97,−1}
In N = 5 this is the only point needed since its orbit is dense, but N = 10 has three invariant regions so we need the two symmetric points p2 ≈{-1.0131106564684930952594 76, -0.5} and p3 ≈ {-1.1135163644116067351943, 0.19098300562505257587};

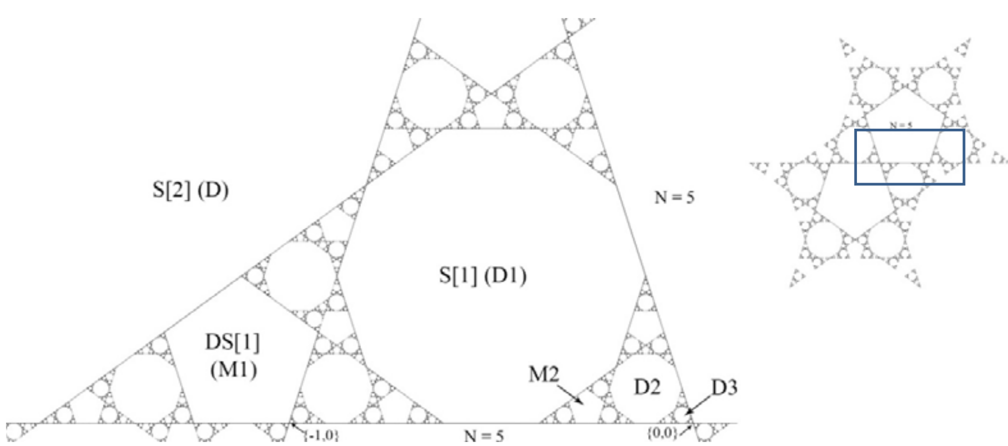


OrbitP1 = V[p1, 50000]; OrbitP2= V[p2,50000]; OrbitP3 = V[p3,50000]

Graphics[{AbsolutePointSize[1.0], Black, Point[OrbitP1], Blue, Point[OrbitP2], Magenta, Point[OrbitP3]}]

The canonical convergence at star[1] of N is now based on M[k] rather then D[k] as with N = 5. Of course they are both dense so either will cover W in the limit. For N = 5, we showed the convergence at star[1] of N = 5 as reproduced below

By convention the N = 5 edge convergence will be studied at star[1] of N as shown here. This is primarily a D[k] convergence but of course the M[k]'s must converge to the origin at star[1] as well.



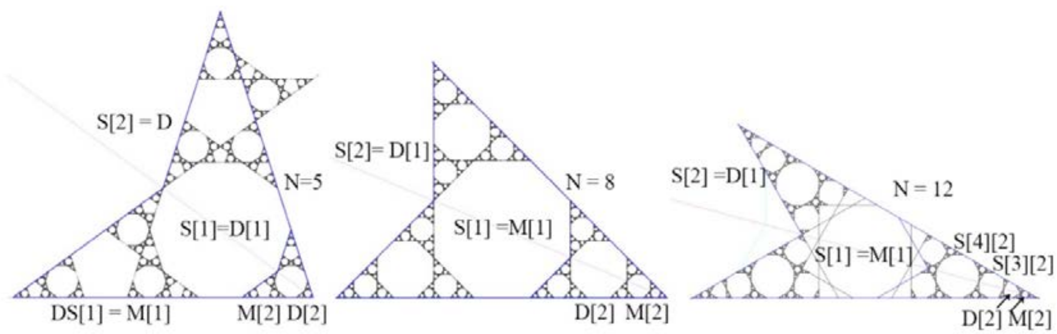
By the First Family Theorem the geometric scaling of these tiles is x = GenScale, so hM[k]/hN = x^k and hD[k]/hD = x^k. To find the fractal dimension it is also necessary to know the τ-period of these tiles. In general this is not a simple issue but the 4k+1 conjecture of [H5] predicts that the ratio of periods will be N + 1 and this can be verified using simple difference equations for pentagons and decagons in the ‘dart’ above between N and D.

Generation	decagons - d_n	pentagons - p_n
1	1 (D1)	2 (M[1] & matching one above it)
2	$7 = 3d_1 + 2p_1$	$10 = 6d_1 + 2p_1$
3	$41 = 3d_2 + 2p_2$	$62 = 6d_2 + 2p_2$
n	$d_n = 3d_{n-1} + 2p_{n-1}$	$p_n = 6d_{n-1} + 2p_{n-1}$

These same difference equations apply to the whole web with initial conditions 5 and 10 so the global solution is $d_n = \frac{5}{7}[8\cdot 6^{n-1} + (-1)^n]$ (See [H2]) This gives decagon center periods of 5, 35, 205, 1235,.. and pentagon periods of 10,50,310,…These difference equations show that the ratio of the periods for the D’s (and M’s) approach 6 as in the 4k+1 conjecture. These D’s can be used to ‘cover’ the star region at all scales, so the Hausdorff-Besicovitch fractal dimension of W is Ln[6]/Ln[1/GenScale[5]] ≈ 1.24114.

This same temporal result is obtained in [H5] by the following argument.

N = 5, 8, 10 and 12 have φ(N)/2 = 2, so they have quadratic complexity- where the only non-trivial scale is GenScale[N]. Since the webs are naturally recursive, a single scale should yield a self-similar web and, under this assumption, we will derive the similarity dimension of these webs below – and hence the Hausdorff fractal dimension.

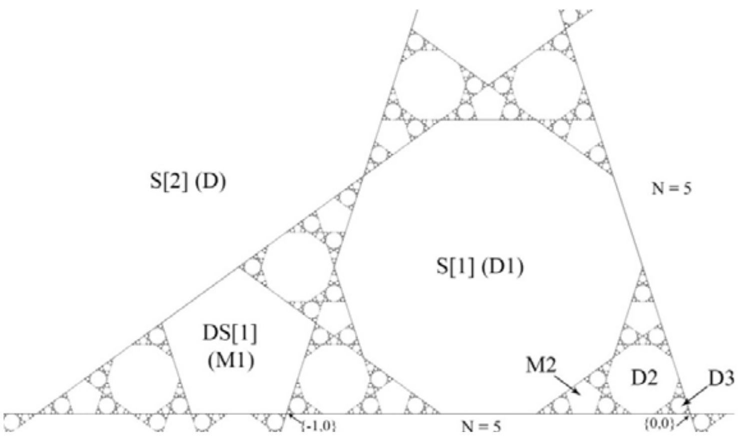


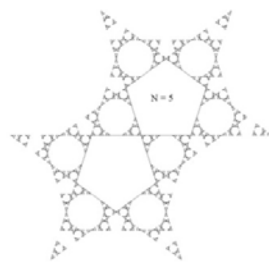
Since the geometric scaling is known, the only issue is the ‘temporal’ scaling – which describes the limiting growth in the number of tiles. In Example 5.1 we used the ‘two dart’ region on the left above to show that the D[k] for N = 5 have a temporal growth factor of 6. Here we will extend this analysis to the ‘three-dart’ structures of N = 8 and N = 12.

For self-similar webs this temporal scaling can be derived from a ‘renormalization’ process – where a representative portion of the web is scaled by GenScale[N] and mapped to itself under τ^k as shown by the magenta lines. As expected, the cases of N = 8 and 12 cases are closely related since their cyclotomic fields are generated by $\{\sqrt{2}, i\}$ and $\{\sqrt{3}, i\}$.

For N = 5 above we used difference equations to obtain the growth factor, but a simple geometric argument relating the M[k] and D[k] will suffice. The blue invariant region consists of two overlapping triangles or ‘darts’ and each dart is anchored by an M[k], so the M[k] scale by 2 with generations, and each M[k] is surrounded by 3 D[k+1]’s so the D[k] scale by 6 with generations. This matches the N + 1 temporal scaling predicted by difference equations, symbolic dynamics and the 4k+1 conjecture.

We can use this same N = 5 web plot to determine the convergence at star[1] of D – which is our surrogate N = 10. In this case there are 3 darts anchored by D[2]’s and in the limit each of these D[2] generates 2 M]2]’s (some of which are only partial here in the 2nd generation) - so the M[k] have a growth factor of 6 –just like the D[k]’s and the fractal dimension is unchanged from N = 5.

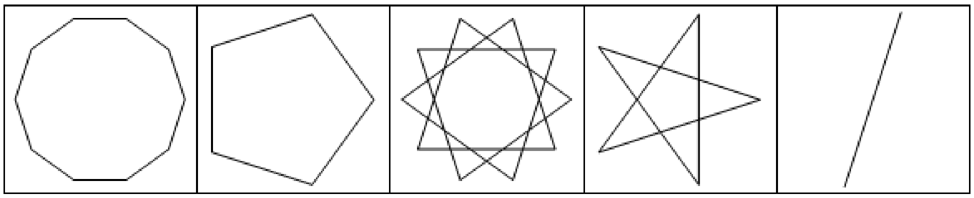




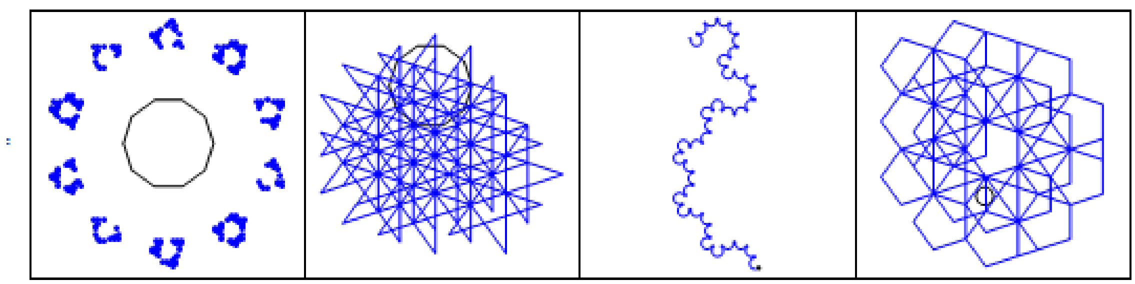
Projections

N = 10 has the same number of non-redundant projections as N = 5 but the extras are interesting so we will do all 4 as shown below:

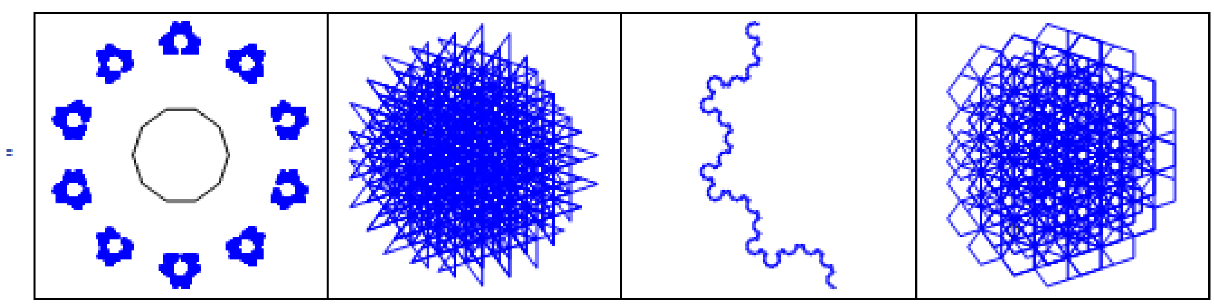
GraphicsGrid[{{Graphics[poly[Wc[[1]]]],Graphics[poly[Wc[[2]]]],Graphics[poly[Wc[[3]]]] ,Graphics[poly[Wc[[4]]]],Graphics[poly[Wc[[5]]]]}},Frame->All]



Example 1**: Ind = IND[p1,50000];** (* Outer ring non-periodic point*)
k = 250; (* the first 250 iterations*)
GraphicsGrid[{{Graphics[{poly[M],Blue,Point[PIM[q1,k,1]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,2]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,3]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,4]]}]}},Frame->All]

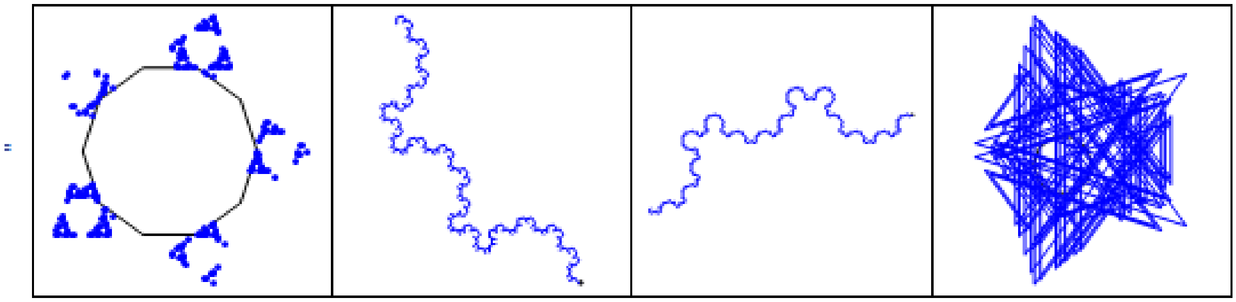


k = 5000; Note that the outer ring is decomposing because P1 is every other point in the orbit. The P3 projection is virtually identical to the N = 5 P2 projection

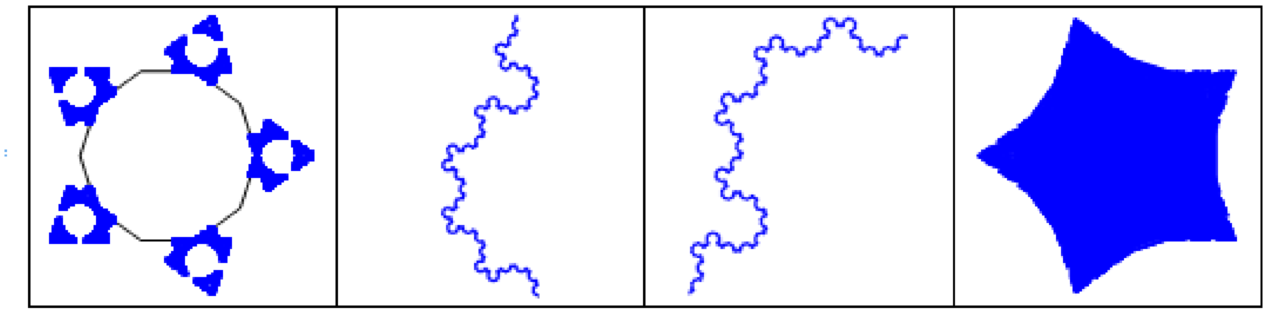


Example 2**: Ind = IND[p2,50000];** (* Inner ring non-periodic point*)

k = 250;



k = 20000;



Below are the first 40 points in the P2 projection of p2 compared with the same points for the P3 projection. D is shown in each for scale.