

Families of Regular Polygons and their Mutations

G.H. Hughes

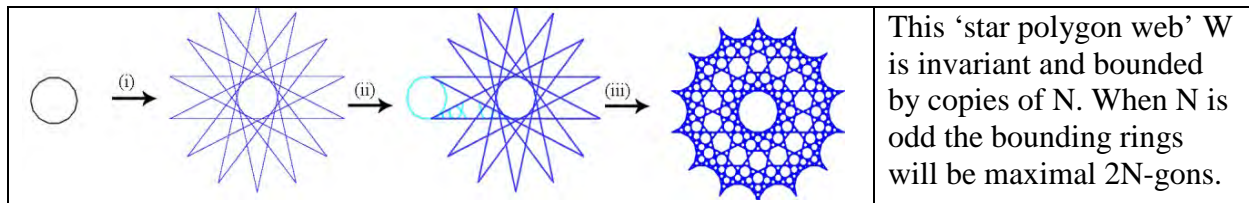
Abstract

Every regular N -gon determines a canonical ‘family’ of regular polygons which are conforming to the bounds of the ‘star polygons’ determined by N . This is illustrated below for the regular tetradecagon known as $N = 14$. These classical star polygons are formed from truncated extended edges of the N -gon and the intersection points (‘star points’) define the parameters of the cyan family of ‘conforming’ polygons. For an N -gon with unit height the horizontal displacements of these star points are of the form $s_k = \tan[k\pi/N]$. These ‘rational multiples of trigonometric functions’ have long been of interest to mathematicians. In 1949, C.L Siegel communicated to S. Chowla a proof that the ‘primitive’ cotangent s_k with $\gcd(k,N) = 1$ are independent when N is prime. In [Ch](1970) Chowla generalized this to any N and in [Gi] (1997) Girstmair showed that it was true for the tangent s_k .

The first three sections of this paper develop the scaling and geometry of these families and in Section 3 we use the Siegel-Chowla result to show that the primitive scales s_1/s_k are a unit basis for \mathbb{Q}_N^+ , the maximal real subfield of the cyclotomic field \mathbb{Q}_N . This is what we call the ‘scaling field’ for N . The traditional generator is $\lambda_N = 2\cos(2\pi/N)$ so it has order $\phi(N)/2$ where ϕ is the Euler totient function. This is also known as the ‘algebraic complexity’ of N . Using the scales as generators yields more meaningful polynomials than the generic polynomials in λ_N .

In Section 4 we introduce the outer-billiards map τ and define the ‘singularity set’ W by iterating the truncated extended edges of N . Since τ is a piecewise isometry, W has the same dihedral symmetry as N and we call it a ‘generalized star polygon’. The limiting ‘web’ W is invariant and bounded by rings of maximal N -gons or $2N$ -gons. Concentric rings guarantee global stability.

The evolution of the singularity set W for the regular tetradecagon known as $N = 14$.



The topology of W is a major concern of this paper, but we will often have to settle for geometry since the dynamics are not well understood. W is the discontinuous analog of a ‘phase space’ for a Hamiltonian system such as our solar system. There are still unanswered questions about the continuous case, but recently there is a consensus among astronomers that our solar system suffered from at least one major instability. (This does not contradict the classical KAM Theorem of 1962 which only showed that there is non-zero measure of initial conditions which will be stable.) At his time the only known cases where the τ - topology is understood are the N -gons with linear or quadratic algebraic complexity – namely $N = 3,4,5,6,8,10$ and 12 . These cases will be described here, and [H6] and [H7] will extend the survey up to $N = 50$. There appears to be an ‘8-fold way’ that defines 8 classes of geometry local to N . See Table 4.4.

Below is a summary of the 5 sections of this paper followed by a historical note. Click on any image to download a higher-resolution version from dynamicsofpolygons.org.

Organization of the five sections of this paper

Section 1 (Star Polygons and Star Points) introduces star polygons and their scaling based on ratios of star points. The main results are the Scaling Lemma relating the scaling of N/k to the scaling of N and the Two-Star Lemma for construction of regular polygons.

Section 2 (Conforming Regular Polygons) shows how the $S[k]$ tiles which form the First Families can be constructed from the $star[k]$ points of N . The main results are the First Family Theorem (FFT) and a generalized version (GFFT) for families of the $S[k]$ based on the premise that the $S[k]$ evolve in a multi-step fashion under the outer-billiards map.

Section 3 (Scaling) shows that the $S[1]$ tile of an N -gon can be used as a scaling reference since $hS[1]/hS[k] = scale[k]$ by the First Family Scaling Lemma. We show that the minimal GenScale is inherent in star polygons so every regular N -gon has the potential to support ‘generations’ of First Families on the edges of its maximal D tile. The Scaling Field Lemma shows that the primitive $scale[k]$ (with $\gcd(k,N) = 1$) are unit generators of the scaling field \mathbb{Q}_N^+ .

Section 4 (The Outer Billiards Map) relates this geometry and scaling to the outer-billiards map τ . Lemma 4.1 uses a height/radius duality between the $star[k]$ points and $S[k]$ centers to show that each $S[k]$ defines a periodic step- k orbit. Example 4.3 shows how the evolution of the web W can be reduced to a ‘shear and rotation’ - which implies that the $S[k]$ evolve in a multi-step fashion. This explains mutations and also justifies the need for a Generalized FFT to characterize the next generation. But there is no clear path for recursion except for the $8k+2$ family where the tiles local to $S[2]$ have a well-defined ‘temporal’ scaling. In general the Edge Conjecture makes predictions about the mod-4 or mod-8 tiles of $S[2]$ which survive in the 3rd generation. This conjecture distinguishes just 8 classes of regular N -gons of the form $8k + j$.

Section 5 (Examples of Singularity Sets) has annotated examples of singularity sets. Appendix I relates the ‘digital-filter’ map of Chau & Lin and ‘dual-center’ map of Erik Goetz to the outer-billiards map. Both of these maps reduce to a ‘shear and rotation’ and their local webs appear to replicate W . Appendix II discusses the large-scale web evolution which is dominated by rings of maximal ‘ D ’ tiles. Appendix III lists open questions.

Historical Background In a 1978 article called “*Is the Solar System Stable?*” Jurgen Moser [Mo2] used the landmark KAM Theorem – named after A. Kalmogorov, V. Arnold and Moser – to show that there is typically a non-zero measure of initial conditions that would lead to a stable solar system. Since the KAM Theorem was very sensitive to continuity, Moser suggested a ‘toy model’ based on orbits around a polygon – where continuity would fail. This is called the ‘outer-billiards’ map and in 2007 Richard Schwartz [S1] showed that if the polygon was in a certain class of ‘kites’, orbits could diverge and stability failed. The special case of a regular polygon was settled earlier in 1989 when A. Shaidanko and F. Vivaldi [SV] showed that all orbits are bounded. The author met with Moser at Stanford University in 1991 to discuss the ‘canonical’ structures that always arise in the regular case - and Moser suggested that a study of these structures would be an interesting exercise in ‘recreational’ mathematics, and this is what it became over the years. However it was soon clear that this discontinuous case retains many of the issues that plagued the authors of the KAM Theorem - namely the topology of the phase space which is what we call the web W . See A3 and Chronology.

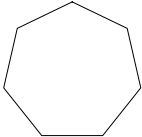
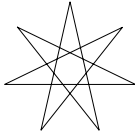
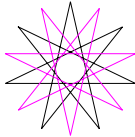
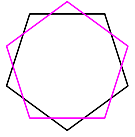
Section 1. Star Polygons and Star Points

‘Star’ polygons or ‘stellated’ polygons were first studied by Thomas Bradwardine (1290-1349), and later by Johannes Kepler (1571-1630).

The vertices of a regular N -gon with radius r are $\{r\cos[2\pi k/N], r\sin[2\pi k/N]\}$ for $1 \leq k \leq N$. A ‘star polygon’ $\{p,q\}$ generalizes this by allowing N above to be rational of the form p/q so the vertices are given by $\{p,q\} = \{r\cos[2\pi kq/p], r\sin[2\pi kq/p]\}$ for $1 \leq k \leq p$.

Using the notation of H.S.Coxeter [Co] a regular heptagon can be written as $\{7,1\}$ (or just $\{7\}$) and $\{7,3\}$ is a ‘step-3’ heptagon formed by joining every third vertex of $\{7\}$ so the exterior angles are $2\pi/(7/3)$ instead of $2\pi/7$.

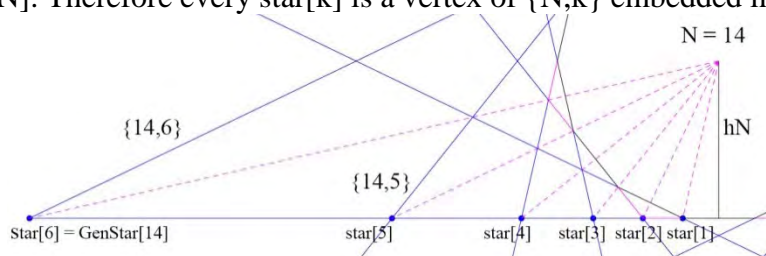
By the definition above, $\{14,6\}$ would be the same as $\{7,3\}$, but there are two heptagons embedded in $N = 14$ and a different starting vertex would yield another copy of $\{7,3\}$, so a common convention is to define $\{14,6\}$ using both copies of $\{7,3\}$ as shown below. This convention guarantees that all the star polygons for $\{N\}$ will have N vertices.

$\{7,1\}$ (a.k.a. $N = 7$)	$\{7,3\}$	$\{14,6\}$	$\{10,2\}$
			

The number of ‘distinct’ star polygons for $\{N\}$ is the number of integers less than $N/2$, which we write as $\langle N/2 \rangle$. So for a regular N -gon, the ‘maximal’ star polygon is $\{N, \langle N/2 \rangle\}$.

Our default convention for the ‘parent’ N -gon will be centered at the origin with ‘base’ edge horizontal, and the matching $\{N,1\}$ will be assumed equal to N . In general s_N , r_N and h_N will denote the side, radius and height (apothem). Typically we will use h_N as the lone parameter.

Definition 1.1 The *star points* of a regular N -gon are the intersections of the edges of $\{N, \langle N/2 \rangle\}$ with a single extended edge of the N -gon (which will be assumed horizontal). By convention the star points are numbered from $\text{star}[1]$ (a vertex of N) outwards to $\text{star}[\langle N/2 \rangle]$ - which is called $\text{GenStar}[N]$. Therefore every $\text{star}[k]$ is a vertex of $\{N,k\}$ embedded in $\{N, \langle N/2 \rangle\}$.



The star points could equally be defined on the positive side of N , but this ‘left-side’ convention is sometimes convenient. The symmetry of these choices makes it irrelevant which convention is used. In general, the star points of a regular N -gon with apothem h_N have coordinates:

$$\text{star}[k] = \{\pm h_N \cdot s_k, -h_N\} \text{ where } s_k = \text{Tan}[k\pi/N] \text{ for } 1 \leq k < N/2.$$

The *primitive* $\text{star}[k]$ and s_k are those with $(k,N) = 1$, where (k,N) is $\text{gcd}(k,N)$. Sometimes we will loosely refer to the s_k as ‘star’ points.

Note: In the Introduction we noted that there is a long history of interest in trigonometric functions of rational multiples of π . Niven [N] shows that these ‘trigonometric numbers’ are also algebraic numbers. When C.L. Siegel communicated to S.Chowla [Ch] a proof that the primitive s_k for the cotangent function are linearly independent, he only proved it for prime N . In 1970 Chowla generalized this result using character theory and Dirichlet’s \mathcal{L} -series, but the general tangent case was only settled recently in [Gi] (1997). Therefore the primitive s_k are linearly independent and it follows that any non-primitive s_k must be a \mathbb{Q} -linear combination of the primitive s_k but the coefficients may be very hard to find. See Corollary 3.1 to Lemma 3.3.

Section 3 is devoted to scaling, but here we present the basic definitions and prove the Scaling Lemma. In Section 3 we will use these scales as a basis for the maximal real subfield of \mathbb{Q}_N .

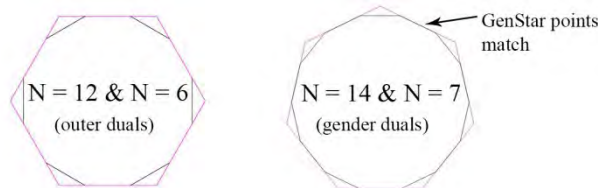
Definition 1.2 The (*canonical*) *scales* of a regular N -gon are $\text{scale}[k] = s_1/s_k$ for $1 \leq k < N/2$. The *co-scales* are of the form s_k/s_1 . The *primitive scales* or *coscales* are those with $(k,N) = 1$. $\text{GenScale}[N]$ is defined to be $\text{scale}[\langle N/2 \rangle]$.

By definition $\text{scale}[1]$ is always 1 and GenScale is the minimal scale. Since these scales are independent of height, to compare scales for an N -gon and an N/k -gon, the later can be regarded as circumscribed about the N -gon with shared center and height – so the sides can be compared.

Definition 1.3 If N and M are regular polygons with $M = N/k$, then $\text{ScaleChange}(N,M) = (s_1 \text{ of } N)/(s_1 \text{ of } M) \leq 1$. This is abbreviated $\text{SC}(N,M)$ or just $\text{SC}(N) < 1/2$, when M is $N/2$.

The case when N and M are related by $M = N/2$ will play an important role in what follows.

Definition 1.4 For N even, a regular $N/2$ -gon is the *outer-dual* of N if the $N/2$ -gon can be circumscribed about the N -gon with matching ‘base’ edge, center and height. In this case $s_N/s_{(N/2)}$ will be $\text{SC}(N) = s_1/s_2 = \text{scale}[2]$ of N . When N is twice-odd, this outer-dual will be called the *parity-dual* or *gender-dual* of N . This is the only case where the GenStars match.



Lemma 1.1 (Scaling Lemma) Suppose N and M are regular polygons and $M = N/k$, then $\text{scale}[j]$ of $N/k = \text{scale}[kj] / \text{scale}[k]$ of N

Proof: There is no loss of generality in assuming that N and N/k are in ‘standard position’ at the origin with equal heights so N/k will be a ‘circumscribed factor polygon’ of N . If s is the side of N , then $s/\text{SC}(N,N/k)$ will be the side of N/k . The external angle of N/k is $2\pi k/N$ so in this position, every k th edge will coincide with an edge of N . Therefore it will share every k th star point with N and by definition the corresponding scales are related by the ratio of the sides of N and N/k , so $\text{scale}[j]$ of $N/k = (\text{scale}[kj] \text{ of } N)/\text{SC}(N,N/k)$. Since $\text{scale}[1]$ of $N/k = 1 = \text{scale}[k]/\text{SC}(N,N/k)$, it follows that $\text{SC}(N,N/k)$ is $\text{scale}[k]$ of N . \square

Therefore $\text{scale}[k]$ of $N = 7$ is the same as $\text{scale}[2k]/\text{scale}[2]$ for $N = 14$ and this twice-odd case is the only nesting where the GenStar points coincide, so $\text{GenScale}[7] = \text{GenScale}[14]/\text{scale}[2]$.

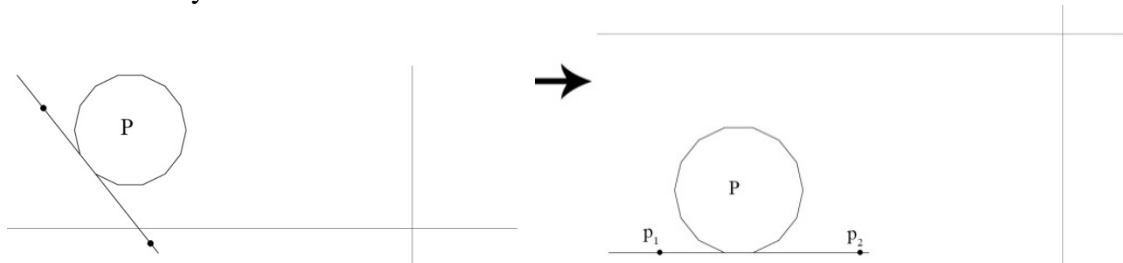
Lemma 1.2 When N is even, $\text{GenScale}[N] = \tan^2[\pi/N] = s_1^2$ and when N is odd $\text{GenScale}[N] = \tan[\pi/2N] \cdot \tan[\pi/N] = s_1 \cdot s_2$ of $2N$.

Proof : For N even, the star points inherit the reflective symmetry of N under $\tan[\pi/2 - \theta] = \cot[\theta]$, so $s_{N/2-k} = 1/s_k$. Setting $k = \langle N/2 \rangle = N/2 - 1$, $\text{GenScale}[N] = s_1/(1/s_1) = s_1^2$. When N is odd the Scaling Lemma says that $\text{GenScale}[N] = \text{GenScale}[2N]/\text{scale}[2] = s_1^2/(s_1/s_2) = s_1 \cdot s_2$ of $2N$. \square

To define a regular N -gon in a given coordinate system, it is sufficient to know its height (apothem) and its center, but both of these are determined by knowing the co-ordinates of two star points, as any cartographer would know.

Lemma 1.3 (Two-Star Lemma) If P is a regular N -gon, any two star points are sufficient to determine the center and height.

Proof. By definition, the star points lie on an extended edge of P . There is no loss of generality in assuming that this extended edge is parallel to the horizontal axis of a known coordinate system with arbitrary center.



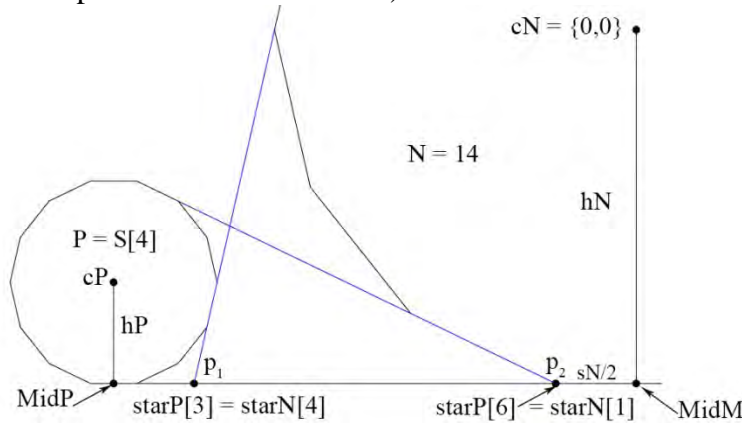
Since all points on this extended edge will have a known second coordinate, we will just need the horizontal coordinates of the star points, namely p_1 and p_2 with $p_2 > p_1$. Therefore $d = p_2 - p_1$ will be positive. Relative to the polygon P , $p_1 = \pm \text{star}[j][[1]] = \pm hP \cdot s_j$ and $p_2 = \pm \text{star}[k][[1]] = \pm hP \cdot s_k$ (where $x[[1]]$ is the horizontal coordinate of x). These indices j and k must be known. There are only two cases to consider:

(i) If p_1 and p_2 are on opposite sides $hP = (p_2 - p_1)/(s_k + s_j)$

(ii) If p_1 and p_2 are on the same side of P , there is no loss of generality in assuming that it is the right-side of P because star points always exist in their symmetric form with respect to P . In this case we can assume that $1 \leq j < k < N/2$ so $hP = (p_2 - p_1)/(s_k - s_j)$

Now that hP is known, the horizontal displacement of p_1 and p_2 relative to the center of P are $x = hP \cdot s_j$ and $d + x = hP \cdot s_k$ if both are on the same side or $x = hP \cdot s_j$ and $d - x = hP \cdot s_k$ if they are on opposite sides. Of course only one of these displacements is needed to define the center of P . \square

Example 1.1 (The one-elephant case) P below shares two star points with the elephant N = 14 which defines the coordinate system. The shared star points are (right-side) star[3] and star[6] of P which we write as starP[3] and starP[6]. (The points p₁ and p₂ shown below are the horizontal coordinates of these points relative to MidM.)



Because of the symmetry between P and N, either one could be used as reference to construct the other, but here we assume that hN is known and it is desired to find hP (and cP) relative to N.

By the Two-Star Lemma
$$hP = \frac{p_2 - p_1}{\tan[6\pi/14] - \tan[3\pi/14]} = hN \frac{\tan[\frac{4\pi}{14}] - \tan[\frac{\pi}{14}]}{\tan[\frac{6\pi}{14}] - \tan[\frac{3\pi}{14}]}$$

Therefore
$$\frac{hP}{hN} = \frac{hS[4]}{hN} = \frac{\tan[\frac{4\pi}{14}] - \tan[\frac{\pi}{14}]}{\tan[\frac{6\pi}{14}] - \tan[\frac{3\pi}{14}]} = \frac{\tan[\pi/14]}{\tan[3\pi/14]} = \frac{s_1}{s_3}$$
 (which is also $s_1 \cdot s_4$)

Therefore $hP \cdot s_3 = hN \cdot s_1$. By definition, the right side is the horizontal displacement of star[1] from MidN, which is $sN/2$. This must be equal to the left side which is the horizontal displacement of starP[3] from MidP. Therefore the horizontal displacement of P from starP[3] must be $-sN/2$. The trigonometric identity above follows from the fact that the displacement from p₁ to p₂ must be the same when viewed from N or from P.

P here is known as S[4] because it was constructed using star[4] (and star[1]) of N. Clearly this same construction can be carried out for the remaining star[k] points of N = 14 and we will do this below for arbitrary N. Since each S[k] will have a symmetric relationship with N, the displacements will always be $-sN/2$ and every S[k] will share its star[N/2-1] point with N.

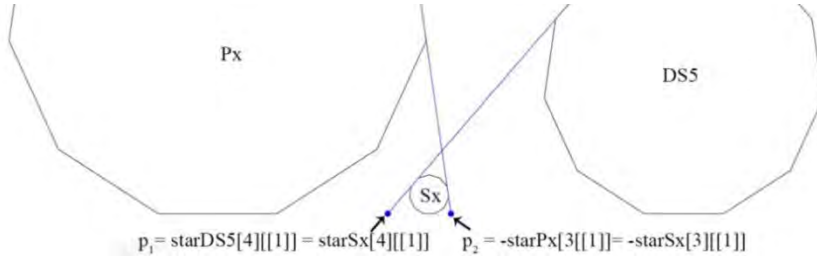
Algebraically the S[k] and N are closely related because hP/hN will always be an element of the ‘scaling field’ defined by N = 14 (or N = 7). This is the number field generated by $2\cos[2\pi/7]$ or $\text{GenScale}[7] = \tan[\pi/7] \cdot \tan[\pi/14]$. See Section 3. Using Mathematica:

AlgebraicNumberPolynomial[ToNumberField[hS[4]/hN, GenScale[7]], x] = $\frac{x^2 - x + 1}{2}$

So $hS[4]/hN = \frac{x^2 - x + 1}{2}$ where $x = \tan[\pi/7] \cdot \tan[\pi/14]$ and this scale is an algebraic number.

Since N = 14 and the matching N = 7 have $\phi(N)/2 = 3$ where ϕ is the Euler totient function, they are classified as ‘cubic’ polygons. This means that any generator of the scaling field will have minimal degree 3, so the scaling polynomials will be at most quadratic.

Example 1.2 (The two-elephant case) The tiles below exist in the coordinate space of $N = 11$. This is a ‘quintic’ N -gon so the algebra is much more complex than $N = 14$. P_x and $DS5$ share a star point which is off the page at the right, but they do not share any other star points so it was a challenge to find a second defining star point of P_x , even though the parameters and star points of $DS5$ are known from the First Family Theorem to follow. Since these two elephants are only distantly related, it is unusual for them to share a third tile, which we call S_x . This S_x tile shares extended edges with both P_x and $DS5$ so it could be used to define both of them.



These tiles share the ‘base’ edge of $N = 11$. The calculations below assume $hN = 1$ so the star points have vertical coordinate -1 and we will just need the horizontal coordinates.

$$p_1 = \text{starDS5}[4][[1]] = -\tan\left[\frac{\pi}{22}\right]^2 \left(2\cot\left[\frac{\pi}{22}\right] - \cot\left[\frac{\pi}{22}\right]^3 + \tan\left[\frac{\pi}{11}\right] \right) + 2\cot\left[\frac{\pi}{22}\right] \left(-1 + 2\tan\left[\frac{\pi}{22}\right]^2 + \tan\left[\frac{\pi}{22}\right]^3 \tan\left[\frac{\pi}{11}\right] \right) - \cot\left[\frac{3\pi}{22}\right] \tan\left[\frac{\pi}{22}\right]^2 \tan\left[\frac{\pi}{11}\right] \left(2 + \tan\left[\frac{\pi}{22}\right] \tan\left[\frac{\pi}{11}\right] \right) \tan\left[\frac{5\pi}{22}\right] + \tan\left[\frac{\pi}{11}\right] \left(1 - \tan\left[\frac{\pi}{22}\right] \left(2 + \tan\left[\frac{\pi}{22}\right] \tan\left[\frac{\pi}{11}\right] \right) \tan\left[\frac{5\pi}{22}\right] \right)$$

The trigonometric expression for $\text{star}[3]$ of P_x is more complex because P_x shares no canonical scaling with N . Mathematica prefers to do these calculations in ‘cyclotomic’ form, which will have vanishing complex part here. The cyclotomic form for $\text{star}[3]$ of P_x is:

$$p_2 = -\text{starPx}[3][[1]] = -\frac{i \left(3 - 8(-1)^{1/11} + 6(-1)^{2/11} + 4(-1)^{3/11} + 11(-1)^{4/11} + 11(-1)^{5/11} + 4(-1)^{6/11} + 6(-1)^{7/11} - 8(-1)^{8/11} + 3(-1)^{9/11} \right)}{\left(1 + (-1)^{2/11} \right)^2 \left(-1 + (-1)^{5/11} \right)}$$

$$\text{By the Two-Star Lemma, } hS_x = \frac{(p_2 - p_1) / (\tan[4\pi/11] + \tan[3\pi/11])}{-\left(2(1i + 2(-1)^{1/22} - 5(-1)^{3/22} - 5(-1)^{5/22} + 2(-1)^{7/22} - 3(-1)^{9/22} - 2(-1)^{13/22} + 2(-1)^{17/22} - 11(-1)^{19/22} + 3(-1)^{21/22} \right)} / \left(\left(5 - (-1)^{1/11} + 6(-1)^{2/11} - (-1)^{3/11} + 5(-1)^{4/11} - 3(-1)^{5/11} + 4(-1)^{6/11} - 4(-1)^{7/11} + 4(-1)^{8/11} - 4(-1)^{9/11} + 3(-1)^{10/11} \right) \left(\cot\left[\frac{3\pi}{22}\right] + \cot\left[\frac{5\pi}{22}\right] \right) \right)$$

The ratio hS_x/hN will be in the scaling field S_{11} which can be generated by $x = \text{GenScale}[11] = \tan[\pi/11] \cdot \tan[\pi/22]$. **AlgebraicNumberPolynomial[ToNumberField[hSx/hN, GenScale[11]]]** $\frac{9}{4} + 52x + \frac{63x^2}{2} + x^3 - \frac{5x^4}{4}$. This is what we call the ‘characteristic’ polynomial for hS_x .

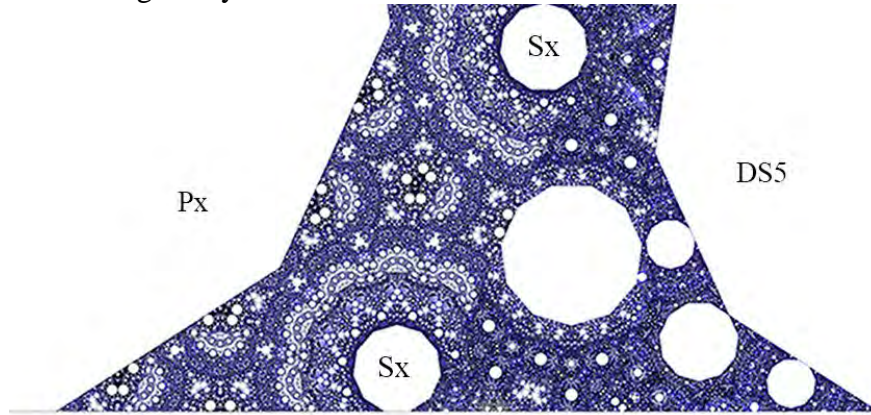
It will be unchanged for any other hN and can also be used for $N = 22$ since $S_{22} = S_{11}$ and the $S[9]$ tile is a surrogate $N = 11$ with known height. Here inside $N = 11$, $hS_x = p(x) \cdot 1 \approx 0.0014424$.

- To construct an exact S_x using p_1 as reference: (i) $\text{MidPointS}_x = \{p_1 + hS_x \cdot \tan[4\pi/11], -1\} \approx \{-6.4628351341351591344, -1\}$ (ii) $cS_x = \text{MidPointS}_x + \{0, hS_x\}$
- (iii) $rS_x = \text{RadiusFromHeight}[hS_x, 11]$ (iv) $S_x = \text{RotateVertex}[cS_x + \{0, rS_x\}, 11, cS_x]$

Under the outer- billiards map τ , the edges of tiles such as S_x are part of the singularity set W , which is also known as the ‘web’. This set will be defined in Section 4. For a regular N -gon it

can be obtained by mapping the extended edges of N under τ or τ^{-1} . Below is a portion of W in the vicinity of S_x . Note that S_x has a clone obtained by rotation about the center of P_x . In the limit this web is probably multi-fractal. See Example 5.4 and [H6]. (Click on the main image or S_x to download larger versions.)

Example 1.3 The singularity set W local to S_x for $N = 11$

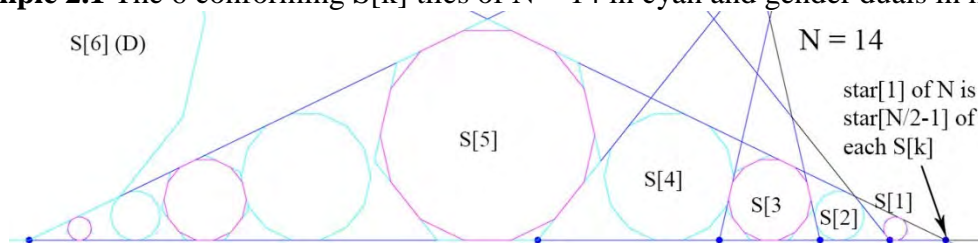


It is our contention that all the polygonal ‘tiles’ (regular or not) which arise from the outer billiards map of N are defined by scales which lie in the scaling field of N . This is the Scaling Conjecture in Definition 3.2. For $N = 11$, knowing the exact parameters of ‘3rd-generation’ tiles like S_x allow us to probe deeper into the small-scale structure of $N = 11$, which is almost a total mystery. But each new generation may involve a significant increase in computational complexity because the star points of each generation depend on the previous, and these have the same complexity as N . So the true computational complexity of S_x may be on the order of $\varphi(N)^3$ and this implies that it may not be feasible at this time to probe deeper than 10 or 12 generations. Since the generations scale by $\text{GenScale}[11] = \tan[\pi/11] \cdot \tan[\pi/22] \approx .042217$ the 25th generation would be on the order of the Plank scale of $1.6 \cdot 10^{-35}$ m. In the words of R. Schwartz, “A case like $N = 11$ may be beyond the reach of current technology.”

Section 2. Conforming Regular Polygons

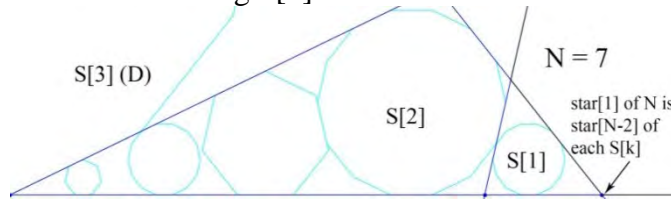
The $S[4]$ tile from Example 1.1 is ‘conforming’ to the bounds of the star polygon of $N = 14$ because it shares the ‘base’ edge of N and the (right-side) GenStar is $\text{star}[1]$ of N as shown below. Clearly there are an infinite number of such conforming N -gons for $N = 14$, but $S[4]$ also shares $\text{star}[4]$ (and $\text{star}[5]$) of N and there are only 6 such ‘strongly conforming’ tiles for $N = 14$. These will constitute the nucleus of the First Family of $N = 14$. It should be clear that only the odd $S[k]$ will have conforming gender duals shown here in magenta.

Example 2.1 The 6 conforming $S[k]$ tiles of $N = 14$ in cyan and gender duals in magenta



When N is odd, the conforming tiles will again be defined relative to the $\text{star}[1]$ point of N as shown below for $N = 7$. But now these conforming $S[k]$ tiles must be $2N$ -gons which share their penultimate star point with N , and they cannot be replaced with their duals while still being conforming (Note that $S[3]$ here can serve as $N = 14$ so the local $S[k]$ are the same as above.) The strongly conforming $S[1], S[2]$ and $S[3]$ tiles of $N = 7$ will be the nucleus of the First Family.

Example 2.2 The 3 conforming $S[k]$ tiles of $N = 7$

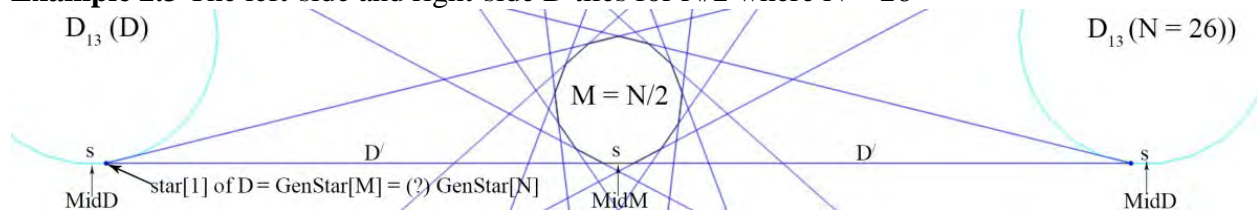


Definition 2.1 (i) If N is even and P is a regular N -gon or $N/2$ -gon, P is *conforming* relative to N if P shares the same base edge as N and GenStar of P is $\text{star}[1]$ of N . (ii) If N is odd and P is a regular $2N$ -gon then P is *conforming* relative to N if P shares the same base edge as N and $\text{star}[N-2]$ of P is $\text{star}[1]$ of N . In both cases P is said to be *strongly conforming* if it is conforming and also shares another star point with N .

Lemma 2.1 (Conformal Replication) Every regular N -gon has a strongly conforming D_N tile which is identical to N for N even and a regular $2N$ -gon with same side as N for N odd.

Proof: Set $\text{star}_{D_N}[1] = \text{GenStar}[N]$ and center offset $sN/2$. When N is even this center and side length defines a regular N -gon D_N which is identical to N . By the reflective symmetry of N , D_N must have $\text{GenStar}[D_N]$ equal to $\text{star}[1]$ of N . When N is odd, again use $\text{GenStar}[N]$ and offset $(sN)/2$ but now construct a regular $2N$ -gon D_N . Because the exterior angle of D_N is half of the exterior angle of N , D_N will have $\text{star}[N-2]$ equal to $\text{star}[1]$ of N so it is strongly conforming. \square

Example 2.3 The left-side and right-side D tiles for $N/2$ where $N = 26$



Lemma 2.2 (Twice Odd Lemma: part 1) Suppose N is twice-odd and M is $N/2$. For M at the origin define D and D_R to be the left and right-side D_M tiles. Then if the origin is shifted to D_R acting as N , D_N will be D , so $N/2$ and N share the same D tiles when N is twice-odd.

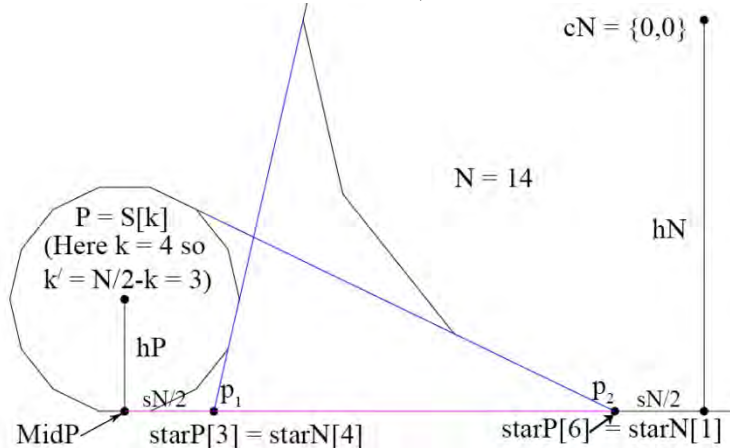
Proof: It is sufficient to show that $\text{GenStar}[M] = \text{GenStar}[N]$. Relative to MidM , $\text{GenStar}[M]$ has horizontal displacement $D' = hM \cdot s_{\langle M/2 \rangle}$ as shown above. Relative to N at the origin $\text{GenStar}[N]$ has horizontal displacement $hN \cdot s_{N/2-1}$ so it is necessary to show that $hN \cdot s_{N/2-1} = 2D' + s/2 = 2hM \cdot s_{\langle M/2 \rangle} + hN \cdot s_1$. Since N is a D tile relative to M , they have the same side so $hM/hN = \text{scale}[2]$ of N , so $hM \cdot s_{\langle M/2 \rangle} = \text{scale}[2] \cdot hN \cdot s_{\langle M/2 \rangle}$ and by definition, s_k of M is s_{2k} of N , so $hM \cdot s_{\langle M/2 \rangle} = \text{scale}[2] \cdot hN \cdot s_{\langle N/2 \rangle} = \text{scale}[2] \cdot hN \cdot s_{N/2-1}$. Therefore we need to show that $s_{N/2-1} = 2 \cdot s_{N/2-1} \cdot \text{scale}[2] + s_1 = 2 \cdot s_{N/2-1} \cdot s_1/s_2 + s_1$. For N even $s_{N/2-1} = 1/s_1$ so this reduces to showing that $2/s_2 + s_1 = 1/s_1$ and this is equivalent to the 'half-angle' formula $s_2 = 2s_1/(1-s_1^2)$. \square

Corollary 2.1 (Corollary to Lemma 2.1) For a regular N-gon, there is a conforming regular N-gon or 2N-gon P with $hP \leq hD_N$.

Proof. By Lemma 2.1, there is a conformal D_M for any regular N-gon M embedded in N which shares the base edge and star[1] of N. This D_M will also be conformal relative to N because N and M have the same exterior angle so a conforming D_M must exist for any $hD_M \leq hD_N$. \square

Theorem 2.1 (First Family Theorem (FFT)) For a regular N-gon every star[k] point defines an S[k] tile which is strongly conforming and has horizontal displacement $-sN/2$ relative to star[k].

Proof: Suppose that N is even with $p_1 = \text{starN}[k]$. Let P be the conforming regular N-gon with center displacement $-sN/2$ relative to p_1 as shown here. Such a P must exist by Corollary 2.1. We will show that P must have $\text{starN}[k]$ as a star point and $\text{starP}[k'] = \text{starN}[k]$ for $k' = N/2 - k$. (k' will be called the 'local index' of P.) We will show that when the displacement of MidP from $\text{starN}[k]$ is $sN/2$ then $k' = N/2 - k$. (Here $k = 4$ so $k' = 3$ so star[4] of N should be star[3] of P.)



Note that there is symmetry between P and N because these two have equal and opposite displacements of $sN/2$ from their respective star points. This implies that $hPs_{N/2-1} = hNs_k$ and defines P as S[k] of N (with $k = 4$ here). If P is known this same formula defines N as S[N/2-1] of P.

Every conforming P has (right-side) $\text{starP}[N/2-1] = \text{star}[1]$ of N, so they all share a global 'index' of $N/2-1$, and here we claim that (right-side) $\text{starP}[k']$ is star[k] of N, so the local index of P is $k' = N/2 - k$. (This 'retrograde' form of the local index is due to the fact that by convention the star points and S[k] are numbered right to left while the star polygon angles increase left to right, so for D at S[N/2-1], $k' = 1$ which corresponds to the minimal star angle of $\phi = 2\pi/N$.)

Any horizontal displacement on the base line can be written from the perspective of N or P and these must match. For the magenta interval above from $\text{starN}[1]$ to MidP, the perspectives are:
 (a) Relative to MidN, it is the displacement of star[k], namely $hN \cdot s_k$
 (b) Relative to MidP this displacement is $hP \cdot s_{N/2-1}$ (always the same form for any conforming P)

$$\text{Therefore } hN \cdot s_k = hP \cdot s_{N/2-1} \text{ so } \frac{hP}{hN} = \frac{s_k}{s_{N/2-1}} = \frac{s_1}{s_{N/2-k}} = s_1 \cdot s_k$$

These last two equalities follow from fact that for N even, $s_{N/2-k} = 1/s_k$ as noted in Lemma 1.2. This implies that $hN \cdot s_1 = hP \cdot s_{N/2-k} = hP \cdot s_k$ and the left side is $sN/2$, so the displacement of MidP from $\text{starP}[k']$ is $sN/2$ and this matches the given displacement of $\text{starN}[k]$. Therefore $\text{starP}[k']$ is $\text{starN}[k]$ of N as claimed. For $k > 1$ all the S[k] will be strongly conforming because they share two distinct star points with N - namely $\text{star}[k]$ and $\text{star}[N/2-1]$ so the theorem is true for N even with the exception of $k = 1$.

When $k = 1$ the local index is the same as the global index but the calculations above are valid by setting $p_1 = p_2$, so $\frac{hS[1]}{hN} = \frac{\text{Tan}[\pi / N]}{\text{Tan}[(N / 2 - 1)\pi / N]} = \frac{s_1}{s_{N/2-1}} = s_1^2$. (Recall that this is $\text{GenScale}[N]$)

Therefore $S[1]$ can be constructed without the help of $\text{star}[2]$, but $S[1]$ always has $\text{star}[2]$ at index 2 because the displacement of $\text{star}[2]$ from $\text{MidS}[1]$ is $hN \cdot (2s_1 - s_2)$ and this is always $hS[1] \cdot s_2 = hN \cdot s_1^2 \cdot s_2$ by the half-angle formula $s_2 = 2s_1 / (1 + s_1^2)$.

Therefore all of the $S[k]$ tiles are strongly conforming and share the same $sN/2$ offset so the First Family Theorem is true for N even. The odd case is a simple doubling of the indices from the even case because the conforming P are now $2N$ -gons with the same $sN/2$ displacements as the even case. Therefore the parameters of the $S[k]$ are:

For N even: $\frac{hS[k]}{hN} = \frac{s_1}{s_{N/2-k}} = s_1 \cdot s_k$ of N

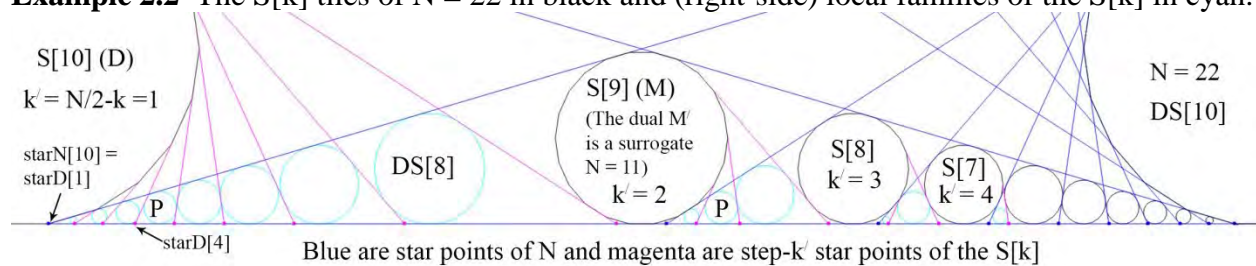
For N odd: $\frac{hS[k]}{hN} = \frac{s_2}{s_{N-2k}} = s_2 \cdot s_{2k}$ of $2N$

This completes the First Family Theorem for arbitrary N . \square

Definition 2.2 below will define the set of $S[k]$ that make up the First Family for arbitrary N . Then we will prove a Generalized First Family Theorem (GFFT) that describes the local families of the $S[k]$. This would not be necessary if these local families were just scaled versions of the First Families, but this is only true in the linear or quadratic cases of $N = 3, 4, 5, 6, 8, 10$ and 12 .

Example 4.3 will show that the $S[k]$ evolve in a multi-step fashion in the web W . The steps will correspond to the local index of $S[k]$ which is $k' = N/2 - k$ for N even or $2k'$ for N odd. This occurs because the 'star angles' formed by the extended edges of N are of the form $k'\phi$ with $\phi = 2\pi/N$. These angles determine the recursive evolution of each $S[k]$. Only the D tiles will have a 'normal' step-1 evolution, as seen below for $N = 22$ where the local family of D is the same as N .

Example 2.2 The $S[k]$ tiles of $N = 22$ in black and (right-side) local families of the $S[k]$ in cyan.

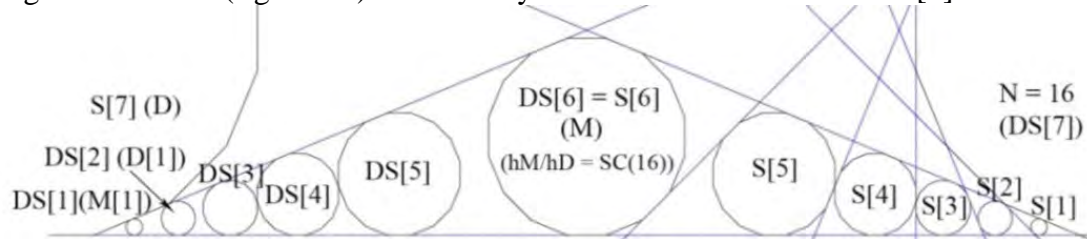


When N is even, $hS[N/2-1]/hN = s_{N/2-1} \cdot s_1 = 1$ so D here is congruent to N , and it is identical to the D_N of the Replication Lemma since it is strongly conforming with displacement $sN/2$ from $\text{GenStar}[N]$. In this N -even case, the central penultimate $S[N/2-2]$ is called M_N or simply M .

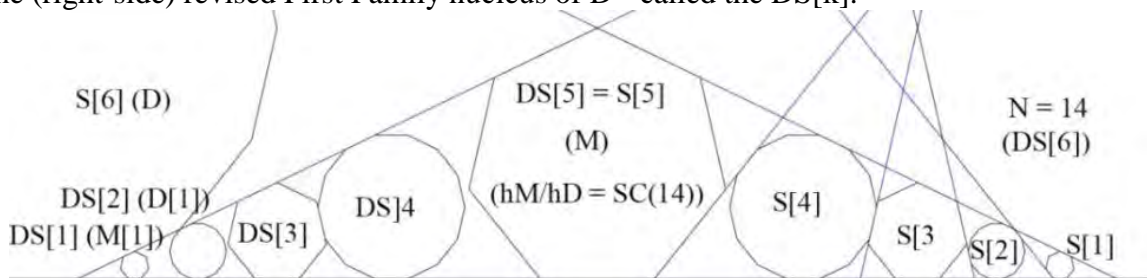
When N is twice-odd M will remain conforming when replaced with the $N/2$ gender-dual M' . As seen in Example 2.1, this 'mutation' can be applied to any odd $S[k]$ (with even k') without violating the conformity of the FFT. M at $k' = 2$ is special because Lemma 2.2 shows that any D tile of N will have a matching $N/2$ tile which will share the same D tiles as N . Since M' and $N/2$ share their left and right side GenStar points, M' can be regarded as a surrogate $N/2$ -gon.

Definition 2.2 (First Family) For any regular N-gon, the First Family Theorem (FFT) defines the strongly conforming $S[k]$ tiles for $1 \leq k \leq \langle N/2 \rangle$. These tiles will be called the First Family Nucleus and as defined earlier the $S[\langle N/2 \rangle]$ tile will be called D and for N even $S[N/2-2]$ will be called M. For $N > 4$, the $S[1]$ & $S[2]$ tiles of D will be called $DS[1]$ & $DS[2]$ or $M[1]$ & $D[1]$.

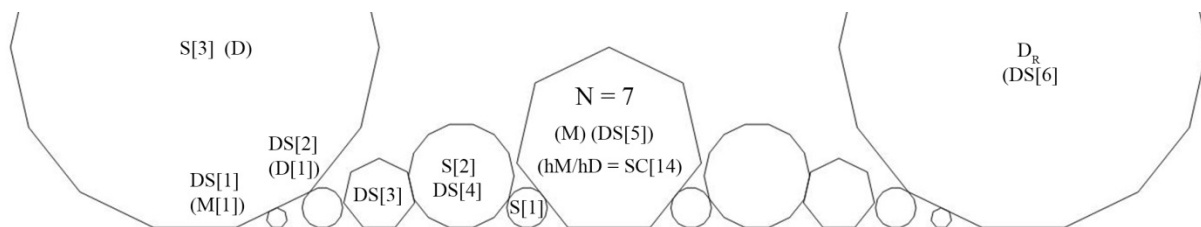
(i) For N twice-even, the First Family will consist of the $S[k]$ in the First Family Nucleus together with the (right-side) First Family Nucleus of D – called the $DS[k]$.



(ii) For N twice-odd, the odd $S[k]$ in the First Family Nucleus will be replaced with their $N/2$ counterparts, and the First Family will consist of this revised First Family Nucleus together with the (right-side) revised First Family nucleus of D - called the $DS[k]$.



(iii) For N odd, the First Family will consist of the First Family Nucleus together with the (right-side) revised First Family Nucleus of D - called the $DS[k]$. We also include the reflections of these tiles about cN . With this convention all First Families span from D to matching D



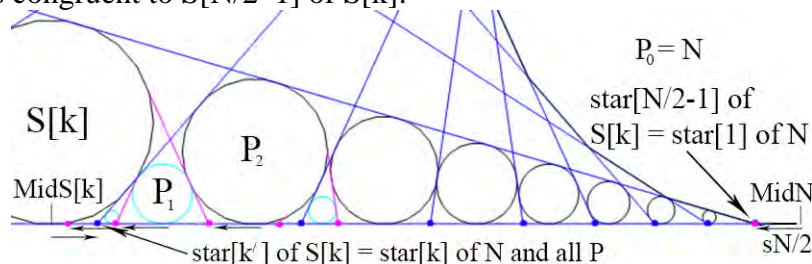
All these families run from D to matching D and involve a fundamental M-D relationship which defines the overall structure of the family. This will define future ‘generations’ (if they exist). In general parts (ii) and (iii) will be identical except for scale. The only issue here is the $S[1]$ tile of $N = 7$ which is technically not part of the First Family of $N = 14$. This tile is regarded as part of the ‘local family’ of $N = 7$ inside $N = 14$. We will define these local families in the GFFT and if desired they can be included as an extended $N = 14$ family. The mapping that transforms any twice-odd N-gon family to the matching $N/2$ -gon family is a simple gender-change scaling followed by a translation of origin from N to M: $T[x] = \text{TranslationTransform} \{ \{0,0\} - cS[N/2-2] \} [x]/SC[N,N/2]$. This mapping and its inverse make up the Twice-Odd Lemma 4.2.

To return to the case of $N = 22$ above, note that the $S[2]$ (P) tile in the local family of M is congruent to the $S[4]$ tile of D or N . It is true in general that the local family of M consists of $S[k]$ which are congruent to $S[2k]$ of N . This is a special case of the GFFT because M has $k' = 2$. For $S[8]$, $k' = 3$, so the local family will be congruent to mod-3 $S[k]$ of N (and by extension this must include the neighbor $S[7]$ as well as N at $S[10]$). In the web, $S[8]$ will look like a normal N -gon because $\gcd(k', 22) = 1$. For $N = 18$, $S[3]$ at $k' = 6$ will be mutated into two interwoven $N/\gcd(k', N)$ -gons at different radii. For N twice-prime the only mutations are when k' is even.

The GFFT below is a direct consequence of the observation made in the FFT about the symmetry between N and the $S[k]$. This implies that for $N = 22$, every $S[k]$ will have N in its local family at $S[10]$ and this step- k' pattern must apply to all local tiles. (We do just the N -even case here.)

Lemma 2.3 (Generalized FFT (GFFT)) For N -even every $S[k]$ defines a ‘family’ of secondary tiles conforming to $\text{star}[k']$ of $S[k]$ where $k' = N/2 - k$. These family tiles will always include N at $S[N/2 - 1]$ and step down by steps of k' . (We show in Section 4 why these are indeed the surviving (right-side) star points of $S[k]$ in the web W .) Therefore the effective star points of $S[k]$ will be subsets of the normal star points of $S[k]$ of the form $\text{star}[k'']$ where $k'' = (N/2 - 1) - j \cdot k'$ for $j \geq 0$ and the corresponding tiles will be congruent to $S[k'']$ of N with offset $sN/2$ from $\text{star}[k'']$ so the $k' = 1$ case will reproduce the (right-side) $S[k]$ of D from the FFT.

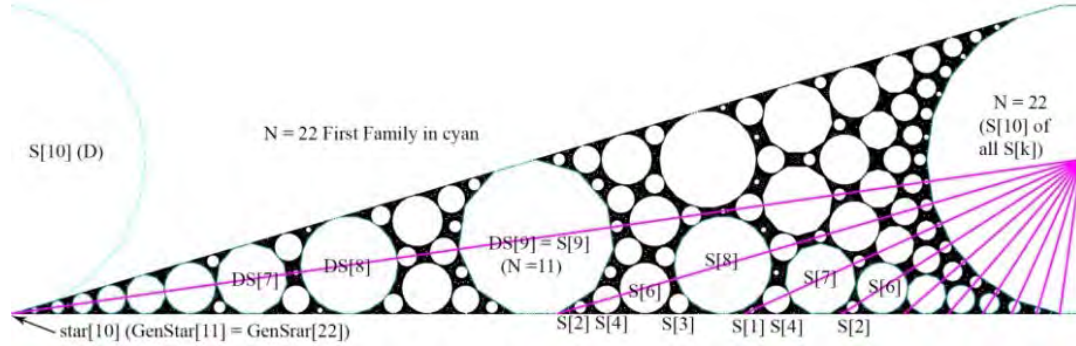
Proof: Suppose $S[k]$ is at the origin with $\text{star}[N/2 - 1]$ which is $\text{star}[1]$ of some P_0 tile and conversely this P_0 tile is conforming to $\text{star}[k']$ of $S[k]$. Then if P_0 and $S[k]$ have equal (and opposite) displacements from their respective star points (as in the FFT), $hS[k] \cdot s_{N/2-1} = hP_0 \cdot s_k$ and $hS[k] = hN \cdot s_1 \cdot s_k$ by the FFT, so $hP_0 \cdot s_k = hN \cdot s_1 \cdot s_k \cdot s_{N/2-1}$ and $hP_0/hN = s_1 \cdot s_{N/2-1} = 1$. Therefore $P_0 = N$ and N is congruent to $S[N/2 - 1]$ of $S[k]$.



This argument will remain valid for conforming P_k iff P_k and $S[k]$ have equal (and opposite) displacement from their respective starpoints. For $S[k]$ the midpoint displacement from $\text{star}[k']$ is defined to be $sN/2$, so any hypothetical P_k must have the opposite displacement from the defining $\text{star}[k'']$ point of $S[k]$. Suppose that $\text{star}[k'']$ is an effective star point of $S[k]$. The conformal replication lemma says that there is a regular N -gon P_k conforming to $\text{star}[k']$ of $S[k]$ with displacement $sN/2$ from $\text{star}[k'']$ as shown above. Therefore $hP_k \cdot s_k = hS[k] \cdot s_{k''} = hN \cdot s_1 \cdot s_k \cdot s_{k''}$ so $hP_k/hN = s_1 \cdot s_{k''}$ and by the FFT, P_k is congruent to $S[k'']$ of N . \square

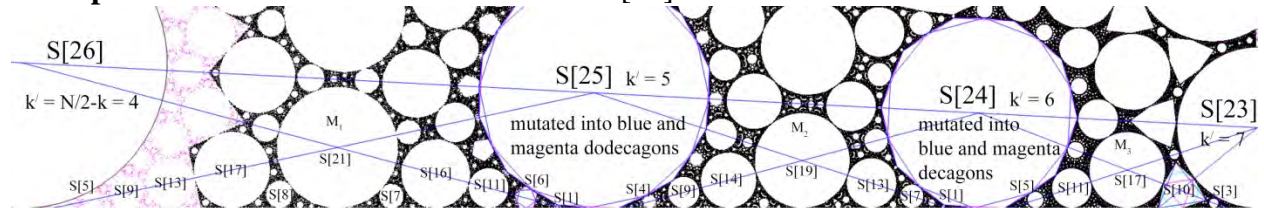
Setting $j = 1$ in the GFFT, $k'' = k - 1$ so every $S[k]$ has the right-side neighbor $S[k-1]$ in its GFFT local family. Here we have concentrated on the right-side families of each $S[k]$ with the understanding that the left-side geometry must match the neighboring $S[k+1]$. All of these predicted secondary $S[k]$ will be conforming to a $\text{star}[k]$ of N , so their centers will be colinear with the origin as illustrated by the magenta lines in the web plots below.

Example 2.3 In Section 4 we we will merge these First Families and secondary GFFT families with the ‘web’ of N . This is a portion of the web for $N = 22$ showing the GFFT families of $S[k]$.



Since N is twice-odd the penultimate $S[9]$ ‘M’ tile can serve as a scaled copy of $N = 11$ and we do not need an ‘odd’ version of the GFFT. Since $k' = N/2 - k = 2$, $S[9]$ will have a step-2 family counting down from N at $S[10]$. It is no coincidence that these are also the $S[k]$ predicted by the FFT. In the same fashion, the $S[8]$ tile of $N = 22$ will have a step-3 right-side family, but the left-side family must be consistent with the right-side of $S[9]$. Here a copy of $S[6]$ is shared by $S[8]$ and $S[9]$ and this determines the $S[3]$ at the base of $S[8]$ to complete the step-3 left-side family.

Example 2.4 The local GFFT families of the $S[25]$ tile of $N = 60$



As explained in the Introduction to [H6], the invariant region local to $N = 60$ extends out to $S[25]$ and beyond this is a barrier formed by the shared edges of $S[13]$ and $S[17]$ of $S[26]$. See Appendix II. Once again it is sufficient to apply the GFFT to just the right side of the $S[k]$ because neighbors share families. Here $S[25]$ is step $N/2 - k = 5$, so the right-side family counts down mod 5 from N which is congruent to D at $S[29]$, so they have the form $29 - 5j$ and the $j = 2$ case yields the local M_2 tile at $S[19]$. This will be the only tile shared by the $S[25]$ and $S[24]$ families so it serves as reference for the step-6 left-side family of $S[24]$. All these ‘satellite’ $S[k]$ are displaced copies of actual $S[k]$ and the GFFT allows us to calculate their exact location. The remaining M_k ‘penultimate’ tiles play similar roles and sometimes they foster their own families but recursion is very limited and the underlying dynamics are very complex. $S[25]$ and $S[24]$ share similar mutations into blue and magenta dodecagons because $N/\gcd(N, k')$ is 12 in both cases. Since N is twice-even, a First Family tile like $S[25]$ will be an N -gon and the $S[6]$ tile of $S[25]$ will ‘inherit’ the same $k' = 24$ web steps as the $S[6]$ of N , so both will be the ‘weave’ of 2 pentagons with $N/\gcd(60, 24) = 5$. In this sense $S[25]$ can be regarded as an in-situ version of N .

The Step-sequence Conjecture of Section 4 explains how to find the step-sequences (and hence orbits) of these satellite $S[k]$. For example [25] must have constant {25} steps in its τ -orbit with period $60/(\gcd(25, 60)) = 12$. There are 4 steps in the local family to reach $S[24]$, so the $S[4]$ of $S[25]$ will be the closest match at {25, 25, 25, 25, 24} and $S[9]$ will have {25, 25, 25, 24}, etc.

Section 3. Scaling of a Regular N-gon

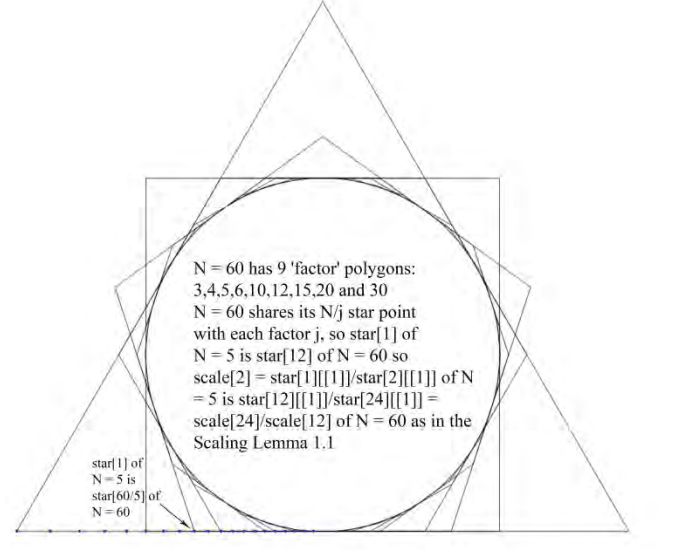
In Section 4 we will introduce the outer-billiards map and prove some basic facts about dynamics, but here we continue to concentrate on the geometry of regular polygons to obtain results that could be applied to any mapping. The most important result of this section will be the Scaling Field Lemma 3.3 which shows that the primitive scales for a regular N-gon will generate the maximal real subfield of the cyclotomic field \mathbb{Q}_N of N. This is what we call the scaling field.

In Section 1 we defined the (canonical) scales of a regular polygon N to be of the form $scale[k] = s_1/s_k$ for $1 \leq k < N/2$ where $s_k = \tan[k\pi/N]$. The primitive scales are those with $\gcd(k,N) = 1$

For N = 60 above, there will be 29 scales and the scaling field will be generated by the $\varphi(N)/2 = 8$ primitive scales so N = 60 is said to have ‘algebraic complexity’ 8. $scale[1]$ is always 1 and the minimal $scale[N/2-1]$ is called $GenerationScale[60] \approx 0.002746$. We will see that this is the scale of (ideal) sequences of S[1] and S[2] tiles converging to $star[1]$ of N (or D).

In this twice-even case it is not clear how the geometry of N = 60 is related to N = 30 or any other factor, but for all factors j, $\mathbb{Q}_{N/j} \subseteq \mathbb{Q}_N$. N has a natural embedding inside each factor polygon that preserves center and height, so the $star[1]$ point of N/j aligns with $star[N/j]$ of N and this makes it easy to compare their scales as in Lemma 1.1 earlier.

Example 3.1 The 9 factor polygons for N = 60

 <p>N = 60 has 9 'factor' polygons: 3,4,5,6,10,12,15,20 and 30 N = 60 shares its N/j star point with each factor j, so $star[1]$ of N = 5 is $star[12]$ of N = 60 so $scale[2] = star[1][[1]]/star[2][[1]]$ of N = 5 is $star[12][[1]]/star[24][[1]] = scale[24]/scale[12]$ of N = 60 as in the Scaling Lemma 1.1</p> <p>$star[1]$ of N = 5 is $star[60/5]$ of N = 60</p>	<p>Since the scales are defined as ratios of the s_k ‘star’ points, any N-gon must share scales with its ‘factors’ because N can be regarded as being embedded in each factor with shared height and center. For any factor k in this embedding, the $star[N/k]$ point of N must be the $star[1]$ point of N/k and Lemma 1.1 says that $scale[j]$ of N/k = $scale[kj]/scale[k]$ of N.</p> <p>This is illustrated here with $k = 12$ and the matching factor polygon N = 5 where $scale[2] = s_1/s_2 = star[1][[1]]/star[2][[1]]$ of N = 5. This must also be $star[12][[1]]/star[24][[1]]$ of N = 60 which is $scale[24]/scale[12]$.</p>
--	---

For N = 30 with factor 2, $scale[j] = scale[2j]/scale[2]$ of N = 60, so $scale[2]$ of N = 60 converts one scale to the other. This $scale[2]$ relationship between N and N/2 applies to the twice-odd case also and there it guarantees that N/2 is a valid member of the First Family of N.

The First Family Theorem defines the heights of the $S[k]$ tiles in terms of the s_k – which we loosely call ‘star’ points.

Lemma 3.1 (First Family Scaling) For all regular N -gons, $hS[1]/hS[k] = \text{scale}[k]$ of N

Proof: For N -even $hS[1]/hS[k] = s_1^2/s_1 \cdot s_k = \text{scale}[k]$ of N . For N odd $hS[1]/hS[k] = s_2^2/s_2 \cdot s_{2k}$ of $2N$ which is by definition $s_1/s_k = \text{scale}[k]$ of N . \square

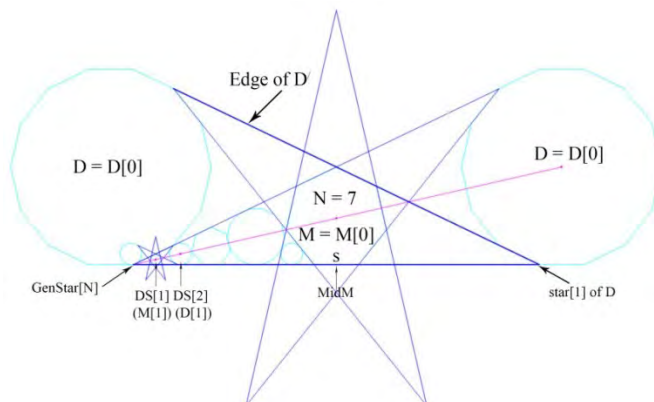
Since D is always the $S[\langle N/2 \rangle]$ tile of N , $hS[1]/hD$ will always be the minimal $\text{scale}[\langle N/2 \rangle]$ which we define to be $\text{GenScale}[N]$. When N is even, D is congruent to N , so $hS[1]/hN = \text{scale}[N/2-1] = \text{GenScale}[N] = s_1^2 = \text{Tan}^2[\pi/N]$. But when N is odd, D is twice-odd so $hDS[1]/hD$ is still s_1^2 of D but it does not make sense to compare N -gons with $2N$ -gons, so our convention is to treat D temporarily as an $N/2$ -gon (with the same side length) by simply dividing by $\text{scale}[2]$ of $2N$. Then $hS[1]/hD = s_1^2/(s_1/s_2) = s_1 \cdot s_2$ of $D = \text{GenScale}[N]$. This will give the correct scaling for both N and D in the twice-odd case while $s_1^2 = \text{Tan}^2[\pi/N]$ may fail to be a unit.

Based on these results, it may seem that $S[1]$ is the natural choice for a ‘next-generation’ tile, but we will discover in Sections 4 and 5 that the most fruitful path to a conforming next generation is by using $S[1]$ and $S[2]$ as a pair and combining their webs together. This will give a combined web which is step-4 relative to $S[2]$ in the even case and step-8 relative to $S[2]$ in the odd case. Both of these webs will have potential to foster next generation tiles on the edges of $S[2]$ playing the role of a second generation $D[1]$. $S[1]$ will still have a largely autonomous local web.

The following Lemma shows that GenScale scaling is inherent in the star polygons because every N -gon can be regarded as an $S[1]$ of some upscaled D' tile. We just do the odd case here.

Lemma 3.2: Every regular N -gon is the $S[1]$ tile of a D' tile where $hN/hD' = \text{GenScale}[D]$

Proof: There is no loss of generality in assuming that D' shares the horizontal edge of N and has $\text{star}[1]$ matching $\text{star}[1]$ of the (right-side) D tile of N , and $sD' = 2hN \cdot s_{\langle N/2 \rangle}$. (i) (N odd) Define D' to be a regular $2N$ -gon. We show in Lemma 2.2 that MidM is the midpoint of the extended edge of D' shown below, so $sN/sD' = 2hN \cdot s_1/2hN \cdot s_{\langle N/2 \rangle} = s_1/s_{\langle N/2 \rangle} \equiv \text{GenScale}[N]$. Since this is a gender mismatch, $hN/hD' = \text{GenScale}[N]/\text{scale}[2] = \text{GenScale}[2N] = s_1^2$ (of $2N$) so by the FFT, N has the correct scale to be the $S[1]$ of D' and by definition it has displacement $sD'/2$ from $\text{star}[1]$, so N is a valid $S[1]$ tile of D' .



The cyclotomic field of N

Definition 3.1 For a regular N-gon, the cyclotomic field \mathbb{Q}_N , is the algebraic number field which can be generated by $\zeta = \exp(2\pi i/N)$ so $\mathbb{Q}_N = \mathbb{Q}(\zeta)$ which can be shown to have order $\varphi(N)$ over \mathbb{Q} . Since $\zeta + \zeta^{-1} = 2\cos(2\pi/N)$, and complex conjugation in \mathbb{Q}_N is always an automorphism of order 2, $\mathbb{Q}(\zeta + \zeta^{-1})$ is order $\varphi(N)/2$. This is called the maximal real subfield and denoted \mathbb{Q}_N^+ . The full field \mathbb{Q}_N is always a quadratic extension of \mathbb{Q}_N^+ . As a vector space, \mathbb{Q}_N is the direct sum of its real and imaginary parts \mathbb{Q}_N^+ and \mathbb{Q}_N^- .

Because of the Lemma below \mathbb{Q}_N^+ is also called the ‘scaling field’ of N – written S_N . Since \mathbb{Q}_N is equivalent to $\mathbb{Q}_{N/2}$ for N twice-odd, $S_N = S_{N/2}$.

Lemma 3.3 (Scaling Field Lemma) For any regular N-gon, the maximal real subfield \mathbb{Q}_N^+ has a unit basis consisting of the primitive (canonical) scales.

Proof: For a given N we have defined $s_k = \tan(k\pi/N)$ for $1 \leq k < N/2$. These ‘star points’ are classified as ‘primitive’ if $(k, N) = 1$ and ‘degenerate’ otherwise. Here we show that the set of primitive scales, $T = \{t_k = s_1/s_k: (k, N) = 1\}$ is a basis for $\mathbb{Q}_N^+ = \mathbb{Q}_N \cap \mathbb{R}$. \mathbb{Q}_N can be generated by any ‘primitive’ N-th root of unity of the form $\zeta^k = \exp(2k\pi i/N)$ with $(k, N) = 1$ so the indices of the primitive scales are also indices of primitive roots of unity.

Since $i \tan(\theta) = (e^{i2\theta} - 1)/(e^{i2\theta} + 1)$, $is_k = (e^{2k\pi i/N} - 1)/(e^{2k\pi i/N} + 1) = (\zeta^k - 1)/(\zeta^k + 1)$ so is_k is in \mathbb{Q}_N and $\text{scale}[k] = \tan[\pi/N]/\tan[k\pi/N] = is_1/is_k = \left[\frac{\zeta - 1}{\zeta + 1} \right] \left[\frac{\zeta^k + 1}{\zeta^k - 1} \right]$ is in \mathbb{Q}_N^+ . We will show that when $(k, N) = 1$, $\text{scale}[k]$ is an algebraic integer in $\mathbb{Z}[\zeta]$ and its inverse $\text{coscale}[k] = \cot[\pi/N]/\cot[k\pi/N]$ is also an integer. Regrouping terms, $\text{scale}[k] = \left[\frac{\zeta - 1}{\zeta^k - 1} \right] \left[\frac{\zeta^k + 1}{\zeta + 1} \right]$. The term on the left is called a

‘cyclotomic unit’ in $\mathbb{Z}[\zeta]$, but we will show from first principles that this product is a unit. Since $(k, N) = 1$, there is a (rational) integer j such that $kj \equiv 1 \pmod{N}$. Therefore

$$\left[\frac{\zeta - 1}{\zeta^k - 1} \right] = \left[\frac{(\zeta^k)^j - 1}{(\zeta^k) - 1} \right] = \sum_0^{j-1} (\zeta^k)^i \text{ and when } k \text{ is odd } \left[\frac{\zeta^k + 1}{\zeta + 1} \right] = \left[\frac{(-\zeta)^k - 1}{(-\zeta) - 1} \right] = \sum_0^{k-1} (-\zeta)^i$$

If k is even N must be odd, so repeat the above with $k' = k+N$. This substitution leaves $\text{scale}[k]$ and $\text{coscale}[k]$ unchanged. Therefore $\text{scale}[k]$ is an algebraic integer.

For $\text{coscale}[k]$ replace ζ with $-\zeta$ so when k is odd the two quotients are $\sum_0^{j-1} (-\zeta^k)^i$ and $\sum_0^{k-1} \zeta^i$.

When k is even, again replace k with $k' = k+N$ which will be odd, and the inverse j can be chosen odd so that it will preserve the sign change. Therefore $\text{scale}[k]$ is an algebraic unit in \mathbb{Q}_N^+ with inverse $\text{coscale}[k]$.

To show that the set T of primitive scales forms a basis for \mathbb{Q}_N^+ , note that $|T| = \varphi(N)/2$ because $(k, N) = 1$ implies that $(N-k, N) = 1$. It only remains to show that the primitive scales are independent over \mathbb{Q} .

Suppose that $\sum_{1 \leq k < N/2} a_i t_k = 0$ with $(k, N) = 1$ and $a_i \in \mathbb{Q}$, then $\frac{1}{s_1} \sum_{1 \leq k < N/2} a_i t_k = \sum_{1 \leq k < N/2} a_i r_k = 0$

where $r_k = \cot[k\pi/N]$ with $(k, N) = 1$. This contradicts the Siegel-Chowla result that the primitive r_k are independent over \mathbb{Q} . Therefore the primitive scales are a unit basis for \mathbb{Q}_N^+ . \square

Example 3.2 ($N = 11$) Set $\zeta = \zeta_{11}$ then $\text{scale}[4] = \frac{\text{Tan}[\pi/11]}{\text{Tan}[4\pi/11]} = \left[\frac{\zeta - 1}{\zeta + 1} \right] \left[\frac{\zeta^4 + 1}{\zeta^4 - 1} \right] = \left[\frac{\zeta - 1}{\zeta^{15} - 1} \right] \left[\frac{\zeta^{15} + 1}{\zeta + 1} \right]$

$$= \left[\frac{(\zeta^{15})^3 - 1}{\zeta^{15} - 1} \right] \left[\frac{(-\zeta)^{15} - 1}{-\zeta - 1} \right] = \sum_{j=0}^2 (\zeta^{15})^j \cdot \sum_{j=0}^{14} (-\zeta)^j = (1 + \zeta^4 + \zeta^8) \cdot \sum_{j=0}^{14} (-\zeta)^j$$

$$\text{coscale}[4] = \frac{\text{Cot}[\pi/11]}{\text{Cot}[4\pi/11]} = \sum_{j=0}^2 (-\zeta^{15})^j \cdot \sum_{j=0}^{14} \zeta^j = (1 - \zeta^4 + \zeta^8) \cdot \sum_{j=0}^{14} \zeta^j$$

This shows a relationship between $\text{scale}[4]$ and ζ^4 which goes back to $\text{star}[4]$ and $S[4]$. Since ζ^4 is a generator of \mathbb{Q}_{11} this relationship can be made exact:

AlgebraicNumberPolynomial[ToNumberField[scale[4], Exp[2*Pi*I/11]^4], x] yields

$\text{scale}[4] = -1 + 2[x^2 + x^3 - x^5 - x^6 + x^8 + x^9]$ where $x = \zeta^4$, Since $S_{11} = \mathbb{Q}_{11}^+$ is generated by $\lambda_{11} = 2\cos(2\pi/11)$ the integers in the field $\mathbb{Q}(\lambda_{11})$ must be in the ring $\mathbb{Z}[\lambda_{11}] = \mathbb{Z}[\zeta + 1/\zeta]$ where $\text{scale}[4]$ simplifies to $3 - 4x - 2x^2 + 2x^3$. Among the scales, the best match for λ_{11} is $\text{GenScale}[11]$ ($\text{scale}[5]$) at $1 - 6x^2 + 2x^4$. We will often use $\text{GenScale}[N]$ as a ‘surrogate’ generator for S_N when scaling tiles. This will yield more meaningful polynomials for the $S[k]$.

Lemma 3.4 (Alternate generators of the scaling field S_N)

(i) For all N , $s_1^2 = \tan(\pi/N)^2$ is a generator of the scaling field S_N

(ii) When N is twice-even, s_1^2 is $\text{GenScale}[N]$ which is a unit generator of S_N .

(iii) When N is odd, $s_1 \cdot s_2$ of $2N$ is $\text{GenScale}[N]$ and it is a unit generator of S_N and S_{2N}

Proof (i) Since $\cos(2\theta) = \frac{1 - \tan^2(\theta)}{1 + \tan^2(\theta)}$ and $\lambda_N = 2\cos(2\pi/N)$ is always a generator of S_N , $\tan(\pi/N)^2$

is also a generator. (ii) When N is twice-even $\tan(\pi/N)^2$ is $\text{GenScale}[N]$ and it is clearly primitive so it is a unit by Lemma 3.3. (iii) When N is odd $\lambda_N = 2/(\text{GenScale}[N] + 1)$ so $\text{GenScale}[N]$ is a generator of S_N and a primitive scale so it is a unit generator of S_N which is identical to S_{2N} . \square

Therefore our convention for (unit) generators of S_N will be to use $\text{GenScale}[N]$ for N twice-even and when N is odd $\text{GenScale}[N]$ will serve as a unit generator for both S_N and S_{2N} . (Any primitive scale of N should also be a generator of S_N but the proof is not obvious.)

The classical paper by J.C. Calcutt [Ca] proves the Complexity Theorem for \cos , \sin and \tan from first principles and develops ‘Chebyshev-like’ polynomials for \tan that highlight the close connection between the s_k and ζ^k . For example $\mathbb{Q}(s_1) = \mathbb{Q}(s_k)$ for primitive s_k and the order of s_1 over \mathbb{Q} is $\varphi(N)$ – except when $4|N$ it is $\varphi(N)/2$. In [H4] we show that the primitive s_k are the positive Galois conjugates of s_1 and $-s_1$. These cases are distinct for $N = 24$. Click to see them.

John Stillwell [St] used results such as these and the half-angle formulas relating \tan to \sin and \cos to say “the *tan* function may be considered more fundamental than either *cos* or *sin*”.

The degenerate scales of N are still in \mathbb{Q}_N^+ so any such scale will be a linear combination of the primitive scales. This should be true for the degenerate s_k themselves and we prove this fact here. Since the s_k are typically not in $\mathbb{Q}(\zeta)$ we will continue to use the is_k as surrogates.

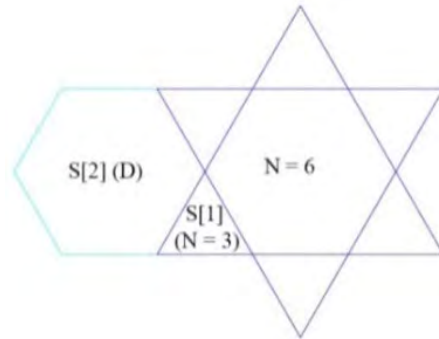
Corollary 3.1 For a given N , every s_j for $j < N$, is a linear combination of the primitive s_k .

Proof: Suppose $(j, N) = m \geq 1$, then $(j/m, N/m) = 1$ and $s_{j/m}$ of $N/m = s_j$ of N . Since $\mathbb{Q}_{N/m} \subseteq \mathbb{Q}_N$, $is_j \in \mathbb{Q}_N^-$ and the set $iS = \{is_k : (k, N) = 1, k < N/2\}$ forms a (vector space) basis for \mathbb{Q}_N^- because $|iS| = \phi(N)/2$ and the is_k are independent over \mathbb{Q} (iff the s_k are independent). Therefore is_j is a linear combination of the primitive is_k which implies that s_j is a linear combination of the s_k . \square

Example 3.3 For a case like $N = 18$, known summation formula such as those found in [P] can be used to find the coefficients. For example $\tan(\pi/18) + \tan(\pi/18 + \pi/3) + \tan(\pi/18 + 2\pi/3) = \tan(6\pi/18) = \sqrt{3}$. Note that all the factors on the left side are primitive in $N = 18$, with k values 1, 7 and 13 ($= -5$), so $\tan(\pi/18) + \tan(7\pi/18) - \tan(5\pi/18) = \sqrt{3}$. In general it can be very difficult to find these coefficients and the same applies to coefficients of the scales.

Even though $s_1^2 = \tan^2(\pi/N)$ will always be a generator of S_N , it may not be a unit or even an integer when N is twice-odd. Therefore it would not be a good choice of generator. The First Family geometry shows clearly the gender issue discussed earlier in this section.

Example 3.4 (The First Family for $N = 6$) The First Family Theorem says that $hS[1]/hN = \text{scale}[2] = \text{GenScale}[6] = \tan^2(\pi/6) = 1/3$, but this is a gender mismatch and not an algebraic integer, so a better choice of scaling is $hS[2]/hN = \text{GenScale}[3] = \tan(\pi/6) \cdot \tan(\pi/3) = 1$. This avoids the gender mismatch and assigns the correct scaling to the web W .



For $N = 3, 4$ and 6 , $\phi(N)/2 = 1$ so they have linear algebraic complexity. This means that there will be no accumulation points in the outer-billiards web W because the web consists of rays or segments parallel to the sides of N and under the outer-billiards map, these segments are bounded apart by linear combinations of the vertices. When N is regular the vector space determined by the vertices has the same rank as \mathbb{Q}_N - namely $\phi(N)$. Therefore the coordinate space of N -gons with linear complexity will have rank 2 over \mathbb{Q} and hence affinely rational coordinates. Since the outer-billiards map is itself an affine transformation these 'rational' coordinates are preserved and the web is also affinely rational - with no limit points.

Definition 3.2 (Canonical polygons) Every regular N-gon defines a coordinate system, and any line segment or polygon P (regular or not) that exists in this co-ordinate system will be called *canonical* relative to N if for every side sP, the ratio sP/sN is in S_N. The Scaling Conjecture says that all tiles and line segments which arise in the web W of the outer-billiards map are canonical.

Example 3.5 Any line segment defined by a linear combination of s_k of N is canonical because if T is such a linear combination T/s₁ is a linear combination of dual scales, so it is in S_N. Therefore the multi-step S[k] mutations in Section 4 will be canonical because the edges are the difference of two star points. Our convention for scaling N-gons is to scale them relative to another N-gon and in this case side scaling is the same as height scaling.

Lemma 3.5 For a regular N-gon, the First Family S[k] tiles are canonical.

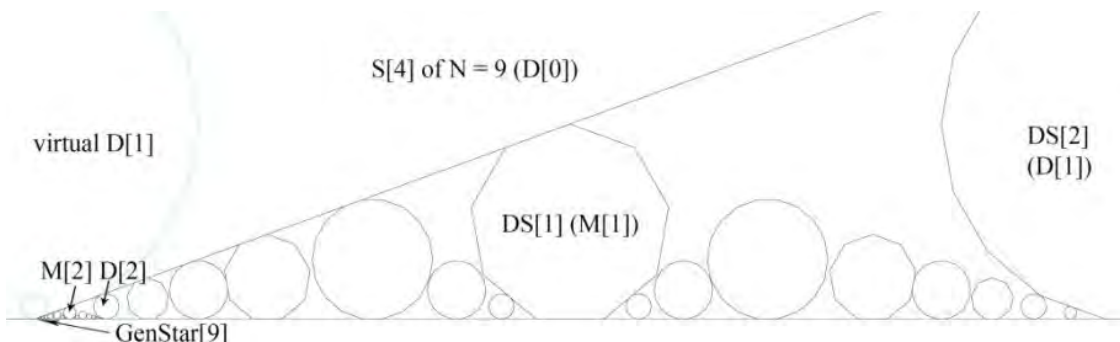
Proof: For N even: $\frac{hS[k]}{hN} = \frac{s_1}{s_{N/2-k}} = s_1 \cdot s_k$ of N - which is scale[N/2-k] of N.

For N odd: $\frac{hS[k]}{hN} = \frac{s_2}{s_{N-2k}} = s_2 \cdot s_{2k}$ of 2N = scale[N-2k]/scale[2] of 2N so it is in S_{2N} = S_N. □

Since the S[k] are canonical regular N-gons or N/2 gons, the (ideal) First Families and subsequent families generated by the S[k] will also be canonical regular N-gons or N/2 gons. The linear or quadratic cases like N = 3,4,5,6, 8, 10, and 12 have webs consisting of only scaled First Family tiles with possible multi-step mutations, so all tiles are canonical.

Definition 3.3 (Ideal generations at GenStar[N]) For any regular N-gon with N > 4, define D[0] = D and M[0] = M (the penultimate tile of D). Then for any natural number k > 0, define M[k] and D[k] to be DS[1] and DS[2] of D[k-1] so for all k, M[k] will be the penultimate tile of D[k]. The (ideal) *kth generation* of N is defined to be the (ideal) First Family of D[k-1]. Therefore M[k] and D[k] will be ‘matriarch’ and ‘patriarch’ of the next generation - which is generation k+1. By Lemma 3.1, hM[k]/hM[k-1] will be GenScale[N], or GenScale[N/2] if N is twice-odd.

Example 3.6 (Ideal generations at GenStar[N] for N = 9, where by convention M[0] = N)



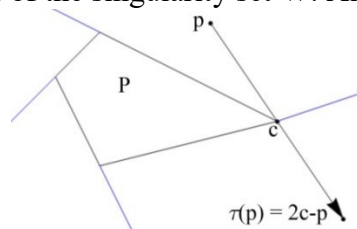
(The Mathematica code for these families is **FFM1 = TranslationTransform[cM[1]][FirstFamily*GenScale]** and **FFM2 = TranslationTransform[cM[2]][FirstFamily*GenScale^2]**.)

Section 4. The Outer-Billiards map

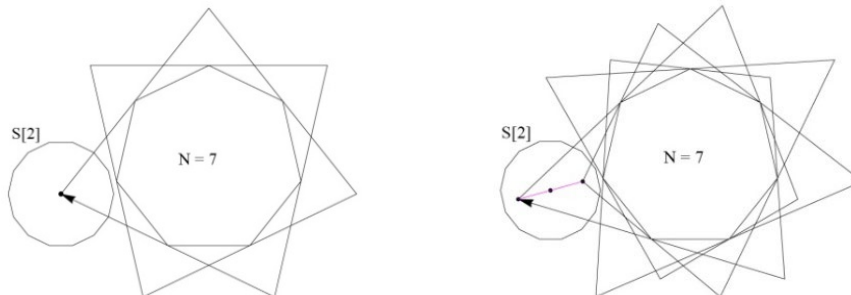
Except for motivation, the geometry of the first three sections is independent of any mapping, and here we will show connections between this geometry and the outer-billiards map τ . In particular we will prove that for any regular N -gon, the First Family $S[k]$ tiles will exist in the complement of the ‘singularity’ set W – which is defined in Definition 4.2 below. This implies that the $S[k]$ arise naturally under iterations of τ . This has long been observed in computer simulations, but now the exact parameters of the $S[k]$ are known. See Lemma 4.1.

In truth it is impossible to properly define τ for ‘most’ regular N -gon because τ is a discontinuous mapping and no one knows exactly where it is defined, because we only know the limiting singularity set W up to approximation. However we do know how W evolves and the $S[k]$ are an integral part of the early evolution of the ‘web’ W .

Definition 4.1 (The outer-billiards map τ) Suppose that P is a convex polygon in Euclidean space with origin internal to P . For an arbitrary point p external to P there will usually be a unique vertex c of P so that the line from p to c can serve as a (cw or ccw) ‘tangent’ line or ‘support’ line. Here we show the clockwise case. If this line is extended an equal distance past c , the new point will be a reflection about c , namely $\tau(p) = 2c - p$. If no such c exists, τ is not defined and p is an element of the singularity set W . All the blue ‘trailing edges’ of P are in W .



Example 4.1 ($S[2]$ for $N = 7$). By Lemma 4.1 to follow, $cS[2]$ will have an orbit that advances two vertices of N on each iteration so it will be period 7. But τ is an inversion so only the center can have odd period. All other points in $S[2]$ will be period 14 as shown in the right. After 14 iterations the original offset from the center will be negated by the second round of 7 iterations.

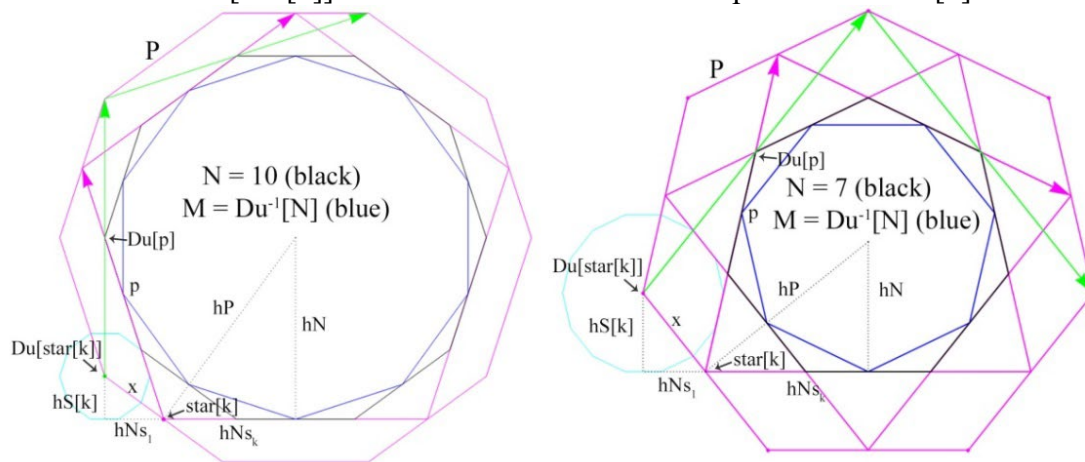


Note that $\tau^k(p)$ will always have the form $2Q + (-1)^k p$ where Q is a sum of k vertices of N with alternating sign. So the center point is period 7 iff $\tau^7(cS[2]) = 2Q - cS[2] = cS[2]$. This has only one solution which is $Q = cS[2]$. For all points in $S[2]$ (including $cS[2]$), any ‘second round’ of 7 iterations will have the same Q (except for sign), so the concatenation yields $Q = 0$ and $\tau^{14}(p) = p$. This is a ‘stable’ period 14 orbit for all points except $cS[2]$.

Lemma 4.1 (Canonical orbits of $S[k]$ centers) For any regular N -gon, $cS[k]$ has an outer-billiards orbit that advances by k vertices on each iteration, so it has period $N/(k,N) \equiv N/\gcd(k,N)$.

Proof: We will show that there is a height/radius ‘duality’ between $cS[k]$ and $star[k]$ defined by $Du[x] = \text{RotationTransform}[-\pi/N, \{0,0\}][x * rN/hN]$. For any N -gon P centered at the origin, $Du[P]$ will map midpoints of edges of P to the adjacent (cw) vertex of P . Therefore Du can be used to map any edge-based (magenta) orbit of a $star[k]$ point to a (green) vertex-based orbit of $Du[star[k]]$ as shown here for $N = 10$ and $N = 7$. (The green orbits are shortened.)

Every $star[k]$ point is a vertex of an $\{N,k\}$ star polygon embedded in $\{N, \langle N/2 \rangle\}$. Therefore this star polygon defines an ‘edge-orbit’ O_k which coincides with the edges of N . This orbit advances k edges on each iteration so it has period $N/(k,N)$. (Since $k = 2$ here the periods are 5 and 7.) Because $\{N,k\}$ inherits the rotational symmetry of N , these magenta orbits will extend equal distance on either side of the midpoint of each edge, so O_k also defines an outer-billiards orbit relative to M -which has vertices equal to the midpoints of N . Since $Du[M] = N$, it follows that $Du[O_k]$ will be an outer-billiards orbit around N with period $N/(k,N)$ as shown in green. Every vertex point in this orbit is $Du[p]$ for some midpoint p in O_k . The initial point of these green orbits must be $Du[star[k]]$ but it is not obvious that this point is also $cS[k]$. We will prove this.



The polygons P were not needed earlier, but they will be useful here. P is defined to be the unique N -gon centered at the origin with $star[k]$ as its midpoint. Therefore $Du[star[k]]$ is a vertex of P and we will show that it must also be $cS[k]$. The length x shown here is $sP/2$ which is hPs_1 (where $s_k = \tan(k\pi/N) = \tan(k\pi/P)$). From the FFT, the displacement of $cS[k]$ from $star[k]$ is $sN/2 = hN \cdot s_1$. Therefore $hS[k]$ should be $s_1 \sqrt{hP^2 - hN^2}$. But $hP^2 = hN^2(1 + s_k^2)$ so this implies that $hS[k] = s_1 \sqrt{hN^2(1 + s_k^2) - hN^2} = s_1 \cdot hN \cdot s_k$. When N is even the FFT says that this is indeed $hS[k]$ but when N is odd $S[k]$ is a $2N$ -gon and $hS[k] = hN \cdot s_2 \cdot s_{2k}$ of $2N$. But s_1 of N is s_2 of $2N$ and s_k of N is s_{2k} of $2N$, so once again $hS[k] = hN \cdot s_2 \cdot s_{2k}$ of $2N$ as in the FFT. Therefore $cS[k]$ is always $Du[star[k]]$ and the period of the orbit of $cS[k]$ will be $N/\gcd(k,N)$. \square

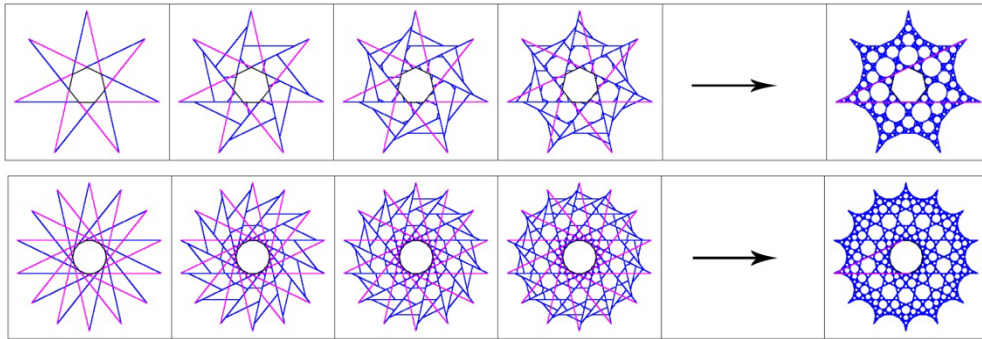
We say that a regular tile P has period k if its center has period k . Of course all points in P must map together but as noted above, when N is even the off-center points may have ‘period doubling’ and visit P twice. This can actually be helpful because it can be used to identify the center. Of course any periodic tile will be in W^C (the complement of W) which has full measure.

Definition 4.2 (The outer-billiards singularity set W of a convex polygon P)

Let $W_0 = \bigcup E_j$ where the E_j are the (open) extended edges of P . The level- k (forward) web is $W_k = \bigcup_{j=0}^k \tau^{-j}(W_0)$ and the level- k (inverse) web is $W_k^i = \bigcup_{j=0}^k \tau^j(W_0)$. $W = \lim_{k \rightarrow \infty} W_k$ and $W^i = \lim_{k \rightarrow \infty} W_k^i$

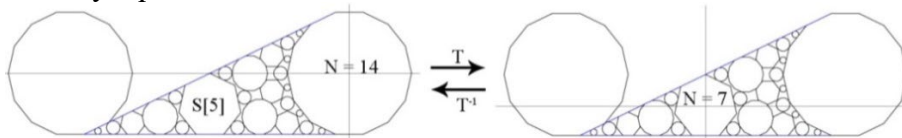
For a regular N -gon, τ^{-1} is τ applied to a horizontal reflection of N , so W_k and W_k^i are also related by a simple reflection and it is our convention to first generate W_k^i by mapping the ‘forward’ extended edges under τ and if desired a reflection gives W_k also. We will assume that in the limit W and W^i must be identical, but at every iteration they will be expected to differ.

Example 4.2 Our default ‘star polygon web W ’ is formed by truncating the extended edges of N where they meet. Below we use $N = 7$ and $N = 14$ as examples and generate the level- k (inverse) webs W_k^i by iterating the blue forward edges under τ^k for $k = 0, 1, 2, 3$ and 50. The magenta trailing edges are shown for reference. At every stage these images could be enhanced by taking the union with the horizontal reflection. In the limit it would not matter.



We call these ‘generalized star polygons’. They retain the dihedral symmetry group \mathcal{D}_N of N . In Appendix II we show that they are invariant under τ because they are bounded by rings of maximal D tiles – which are themselves invariant. In [VS] (1989) rings like these were used to show that the global dynamics are bounded. Our analysis in Appendix II is simplified by the fact that the parameters of these D tiles are known by the FFT. By symmetry the ‘region of interest’ can be restricted further as shown by the magenta First Family regions outlined on the right,

Lemma 4.2 (Twice-odd Lemma) Lemma 2.1 shows that for N twice-odd the First Families of N and $N/2$ are related by the scaling and change of origin between M and N given by $T[\mathbf{x}] = \text{TranslationTransform}[\{0,0\} - \text{cS}[N/2-2]][\mathbf{x}]/\text{SC}[N,N/2]$. Here we show that $N/2$ and N must also have locally equivalent webs.



Proof: Because T and T^{-1} are affine transformations they will commute with the affine transformation τ and hence preserve the web W . \square

Note: In terms of dynamics the N vs. $N/2$ relationship is not trivial but for N even the web steps are $k' = N/2 - k$, and the ratio $\text{Period}[\text{DS}[k]] / \text{Period}[\text{S}[k]]$ will be $k' - 1$ for N even, and for N -odd the D tiles are $2N$ -gons so $k' = 2N/2 - k = N - k$ and the ratio above is now $k' - 2$. Note that by Lemma 4.1, $\text{Period}[\text{S}[k]]$ is $N/\text{gcd}(k,N)$ for all N so $\text{Period}[\text{DS}[k]]$ here is also known.

Note that the $S[k]$ centers of the First Family Theorem are consistent with these shears. Recall that the local index k' of $S[k]$ was defined so that $\text{star}[k]$ of N is $\text{star}[k']$ of $S[k]$. This means that the proposed rotation angle $k'\phi$ is the angle between edge L and the next step- k' edge of $S[k]$.

Therefore rotating the level-0 base edge of $S[k]$ by $k'\phi$ will align a segment of edge L with an edge of $S[k]$. Under τ , this segment will feel the sN shear and form a portion of the 'next' edge of $S[k]$. This is repeated recursively to generate the edges of $S[k]$ in a step- k' fashion.

Any portion of the original level-0 sN segment that extends beyond the bounds of the magenta trailing edges will lie in an adjacent τ -domain and feel a different rotation angle on the next iteration - but the $S[k]$ are conforming to these bounds, so there will always be a portion of the original segment that survives within these bounds and this 'web-cycle' will have period $N/\text{gcd}(k',N)$. By symmetry there must be an M_k cycle on the top edge with this same period.

When N is even $k' = N/2 - k$ so D at $\text{GenStar}[N]$ will have local index $k' = 1$ and the rotation angle will be $\phi = 2\pi/N$ to match that of N . This is the rotation that will allow the hypothetical edge 1 shown above to be a horizontal edge and hence experience the shear sN . The top and bottom shears are ϕ apart because they are on consecutive edges. Therefore for D , the top shear will be relative to edge 12 ($N/2 + 1$). The matching $S[k]$ edges will have this same 12-step orientation with respect to the level-0 horizontal base. This value will be even when N is twice-odd so the primary and secondary cycles are synchronized mod 2 and the $S[k]$ are always formed in a redundant step- k' fashion.

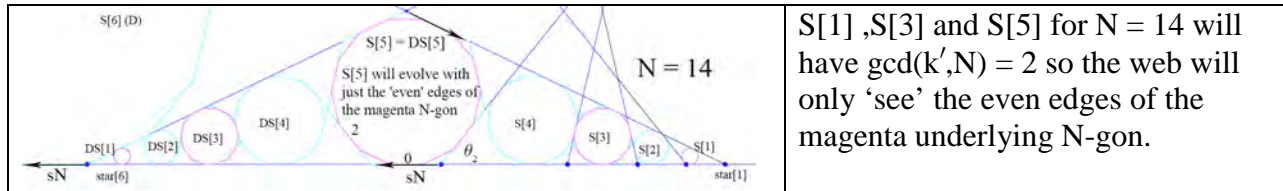
When N is twice-odd, all the odd $S[k]$ will have even k' and hence web-periods $N/2$. Therefore the odd $S[k]$ will be $N/2$ -gons in the web W and this explains the 'androgynous' nature of the twice-odd N -gons. This is what we call a 'mutation'. Whenever $\text{gcd}(k',N) > 1$, the $S[k]$ will be formed from shortened web cycles so $S[6]$ of $N = 18$ will be mutated in a known fashion.

When N is twice-even, $N/2 \pm 1$ is odd and the two web cycles are no longer synchronized mod 2. Therefore mutations will occur iff $\text{gcd}(k',N) > 2$, because when $\text{gcd}(k',N) = 2$ the shortened primary cycle will be corrected by the top cycle which is also period $N/2$. This means that the twice-odd case will have no 'gender' distinction between $S[k]$ for k even or odd. In particular the penultimate $S[N/2 - 2]$ 'M' tile will be free of mutations. Therefore for $N = 12$, $S[2]$ and $S[3]$ will be mutated, but $S[4]$ will not. The $8k+4$ 'family' of N -gons is the only one with mutations in $S[2]$ and this will have a strong influence on future generations. See Example 5.6.

When N is odd the (relative) shears are unchanged from the even case and the star angles are $(\phi/2)(N-2k) = (\phi/2)k'$ where $k' = N-2k$ is double the index for N even. These will be the local indices of the $S[k]$ (which are now $2N$ -gons). Once again D will have index 1 and hence rotation angle $\phi/2$. Therefore D will be a $2N$ -gon with the same side as N . Since $k' = N-2k$ must be odd, the primary web cycle be odd and it will be shortened iff $\text{gcd}(N-2k,2N) > 1$. The top cycle will also be odd since it is based on edge $N+2$ of D , so these two cycles will be synchronized mod 2 as in the twice-odd case, but now both cycles are odd relative to the base edge, so there will be no gender-based mutations in the $S[k]$, but of course D will have the full spectrum of gender changes in the $DS[k]$. In terms of mutations, the $S[3]$, $S[5]$ and $S[6]$ tiles of $N = 15$ will be mutated. See the mutation conjecture below.

Mutations of Regular N-gons

In Example 4.6 we will extend this web analysis to the regions between the $S[k]$ as described in the GFFT but here we summarize what occurs when the step- k' evolution of an $S[k]$ tile has a resonance with N . The simplest example is a resonance of 2 when N is twice-odd. As noted above, the odd $S[k]$ with even k' will have ‘unfinished’ webs which renders them as $N/2$ -gons.



Conjecture for mutations of $S[k]$ due to their multi-step evolution

(i) When N is twice-odd, $S[k]$ will have local index $k' = N/2 - k$ so the rotational period of $S[k]$ in the web will be $N/\gcd(k', N)$ and $S[k]$ will be mutated iff $\gcd(k', N) > 1$.

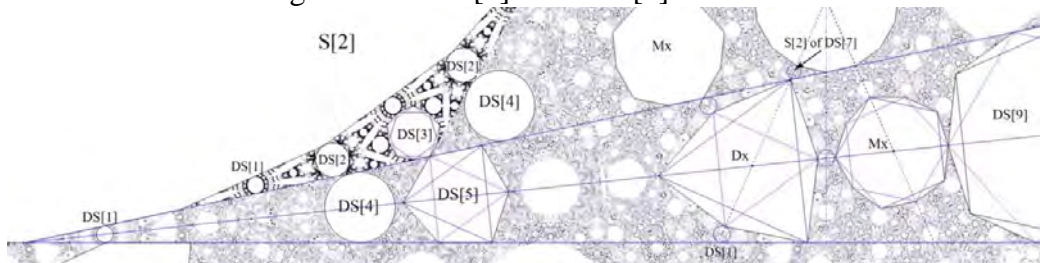
(ii) When N is twice-even the local index of $S[k]$ will be $k' = N/2 - k$ as above so the rotational period is identical, but now it takes an odd number of rotational steps to reach the ‘retrograde’ top edge so the rotational period can be $N/2$ with no mutation of $S[k]$. Therefore $S[k]$ will be mutated iff $\gcd(k', N) > 2$.

(iii) When N is odd, $S[k]$ will have local index $k' = N - 2k$ which is twice the index above because the $S[k]$ are now $2N$ -gons. The rotational period of the $S[k]$ local web will be $2N/\gcd(k', 2N)$ and $S[k]$ will be mutated iff $\gcd(k', 2N) > 1$.

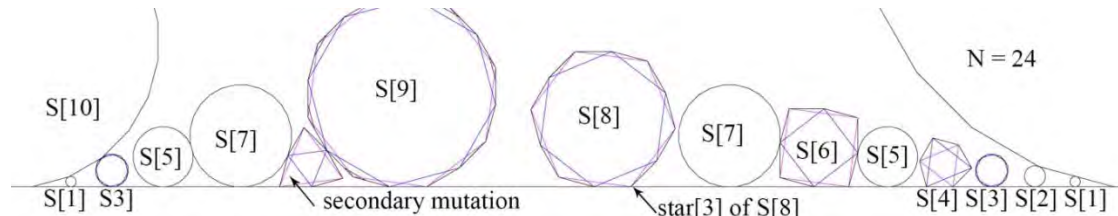
(iv) Typically a mutated $S[k]$ will have a base that spans one web cycle, which is $\gcd(N, k')$ or $\gcd(2N, k')$ star points for N even or N odd. One of these surviving star points should be the minimum of $N/2 - 1 - jk'$ for N even and $N - 2 - jk'$ for N odd. The mutation will be an equilateral $2M$ -gon (or M -gon) where $M = N/(\gcd(k', N))$ for N even or $2N/(\gcd(k', 2N))$ for N odd. \square

In [H6] we extend this conjecture to the next-generation $S[k]$ tiles of $S[2]$ which are called $DS[k]$. When N is even, $S[2]$ seems to inherit the $k' = N/2 - k$ step structure so the $DS[k]$ and $S[k]$ may share the same mutations. For $N = 30$ below $DS[3]$, $DS[5]$ and $DS[9]$ have $k' = 12, 10$ and 6 , so $DS[5]$ is a hexagon while $DS[3]$ and $DS[9]$ are decagons, as in the matching $S[k]$. But when N is odd the $S[k]$ web steps are $N - 2k$ so the mutations may not match.

Mutations in the next-generation $DS[k]$ tiles of $S[2]$ for $N = 30$ in the $8k+6$ family



Example 4.4 (Mutations of $S[k]$ for $N = 24$) Because N is twice-even the web is more forgiving and $S[2]$ and $S[10]$ escape mutation. For $S[8]$, $\gcd(N, k') = 4$, so $\text{MuS}[8]$ will span 4 star points and the minimal (right-side) star point will be as predicted by the GFFT, namely the minimum of $N/2 - 1 - jk' = 3$. This is the smallest star point of the underlying $S[8]$ that survives the local step-4 web and it is an ‘anchor’ for the magenta $N/4$ –gon shown below. The ‘base length’ includes the edge of $S[8]$, so the matching left-side anchor is $\text{star}[1]$ and this will be the reference for the blue hexagon. $\text{MuS}[8]$ will be the ‘interleave’ of these two hexagons, shown here in black.



In [H3] we call $\text{MuS}[8]$ a ‘semi-regular’ dodecagon because it is equilateral and has dihedral symmetry group D_6 rather than D_{12} . The ‘in-vitro’ version at the origin seems to have bounded dynamics but we do not know if this class of semi-regular polygons is always bounded.

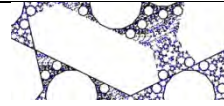
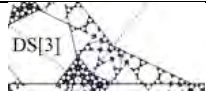
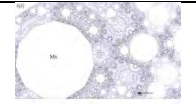


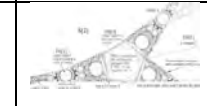
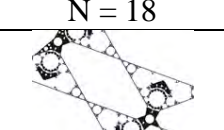
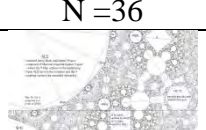
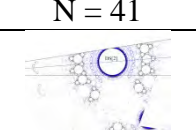

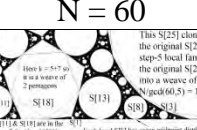
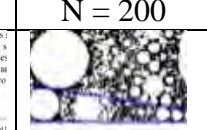
Here are the steps to construct this mutated $S[8]$ based on the known height and center of $S[8]$.
 (i) Find the $\text{star}[3]$ point: $\text{MidS8} = \{cS[8][[1]], -hN\}$; $\text{star}[3] = \text{MidS8} + \{\text{Tan}[3 \cdot \text{Pi}/24] \cdot hS[8], 0\}$;
 (ii) $M1 = \text{RotateVertex}[\text{star}[3], 6, cS[8]]$ (magenta); $M2 = \text{RotateVertex}[\text{star}[1], 6, cS[8]]$ (blue)
 (iii) $\text{MuS8} = \text{Riffle}[M1, M2]$ (black) (This weaves them as in a card shuffle.)

See Example 2.3 for the web. Since these mutations are incomplete local webs, they can be regarded as ‘scaffolding’ for the underlying $S[k]$ tiles. The ideal $S[k]$ will always be in the convex envelope of this scaffolding so in this sense the underlying $S[k]$ are preserved by W .

The general web

Because of the multi-step origin of the $S[k]$ the small-scale web may be very different from the orderly First Family structure. The limiting ‘tiles’ could be points - which must have non-periodic orbits, or possibly lines similar to the structures that appear for $N = 11$ on the right below. No one has ruled out the possibility of limiting regions with non-zero lebesgue measure. This is a long-standing open question in the phase-space geometry of Hamiltonian systems. Any non-limiting tile must be convex with edges parallel to those of N , so it is easy to see that the D tiles are maximal among regular polygons – and in fact rings of these tiles must exist at all radial distances so the dynamics in any finite region must be bounded. See Appendix II.

Example 4.5 Some non-regular tiles in W (click to enlarge)

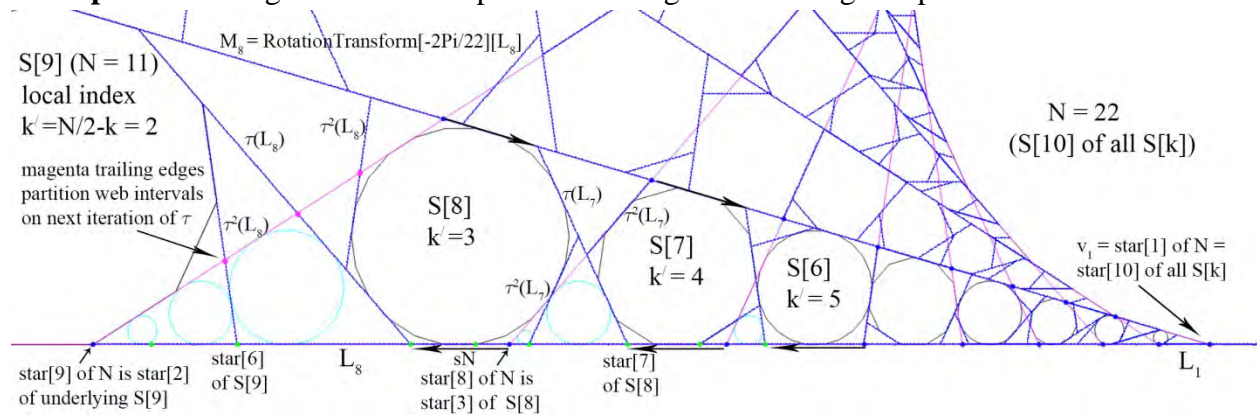
$N = 7$ 	$N = 14$ 	$N = 11$ 	$N = 11$ 	$N = 11$ 	$N = 15$ 
$N = 18$ 	$N = 36$ 	$N = 41$ 	$N = 48$ 	$N = 60$ 	$N = 200$ 

Evolution of the Web : Part II - Local webs of the $S[k]$ and in-situ vs. in-vitro evolution

Every polygon P can be regarded as an ‘in-vitro parent’ at the origin, or P could be just a tile in the web or family of another polygon and we want to know how it evolves ‘in-situ’. When N is twice-odd, the transformation T of Lemma 4.2 relates the in-situ M tile of N with its in-vitro form as an $N/2$ -gon. When N is even the D tile is a retrograde version of N so the in-situ to in-vitro transformation T is a reflection about cM . For most P the two viewpoints are very different.

The previous web analysis in Part I shows that all $S[k]$ (except D) evolve in a multi-step fashion and this will cause visible mutations when the steps are synchronized with N , but even when the final $S[k]$ look normal, the multi-step web evolution will yield star points which are also multi-step and hence the families will be multi-step as shown by the GFFT and the cyan tiles below.

Example 4.6 Enlargement of Example 4.3 showing the surviving star points for $N = 22$



In Example 4.3 the primary interest was the evolution of the $S[k]$ and the mutations that can occur in their web cycles. Here we attempt to describe the local webs of these $S[k]$, because the global web W is by definition the disjoint union of these local webs. In this ‘in-situ’ evolution of the $S[k]$ there is no reason to expect that the right-side geometry will match the left-side, but we can temporarily regard the left-side web of an $S[k]$ to be the right-side web of $S[k+1]$ and concentrate on just the right-side evolution as discussed in Lemma 2.3 (GFFT). The premise of that lemma was that for N even, the ‘effective’ right-side star points of the $S[k]$ were step $k' = N/2 - k$ and Example 4.3 showed that for N even the $S[k]$ evolve in a step k' fashion. This should imply that the surviving star points are also step k' as assumed by the GFFT.

We know that there are two such cycles for each $S[k]$ and each cycle has period $N/\gcd(k',N)$. In the right-side evolution of an $S[k]$, the primary cycle is the ‘top’ cycle which is based on edge $N/2+1$ for N even. It will count down in a skip k' fashion and this explains why the surviving star points are $\text{star}[k'']$ where $k'' = (N/2-1) - j \cdot k'$ for $j \geq 0$. From this it is easy to construct the local cyan $S[k'']$ families as in the GFFT.

Dynamically speaking these families can be regarded as ‘satellites’ of the $S[k]$ which makes them secondary resonances of N . Since the $S[k]$ have constant step- k orbits around N , the step

sequences and periods of the families should reflect this relationship. This is discussed in Appendix C of [H4] and here we give a conjecture about the progression of these sequences.

Step Sequence Conjecture for S[k] families for N even

Suppose the local step- k' ‘GFFT family of S[k] consists of P_j for $j = 1, \dots, m$ where $P_{m+1} = S[k-1]$ and $P_{m+2} = N$ are not part of this immediate family. Then the step-sequence of P_j is periodic of the form $\{(m - j + 1) * \{k\}, k-1\}$ where $*$ is concatenation of copies of $\{k\}$. □

For example with $N = 22$, the local family of S[9] is $P_1, P_2, P_3 = S[2], S[4], S[6]$, so $m = 3$ and the sequence for P_1 has the closest match with S[9] at $\{3 * \{9\}, 8\} = \{9, 9, 9, 8\}$. The sequence for S[4] is $\{9, 9, 8\}$ and finally the local S[6] M’ tile with $\{9, 8\}$ and inherited step $k' = \{11 - 9 + 11 - 8\} = 5$. This can be continued with S[8] which has a local family with $\{8, 8, 7\}$ and $\{8, 7\}$, and S[7] contributes $\{7\}, \{7, 6\}$. After S[7] all the local families contain just N.

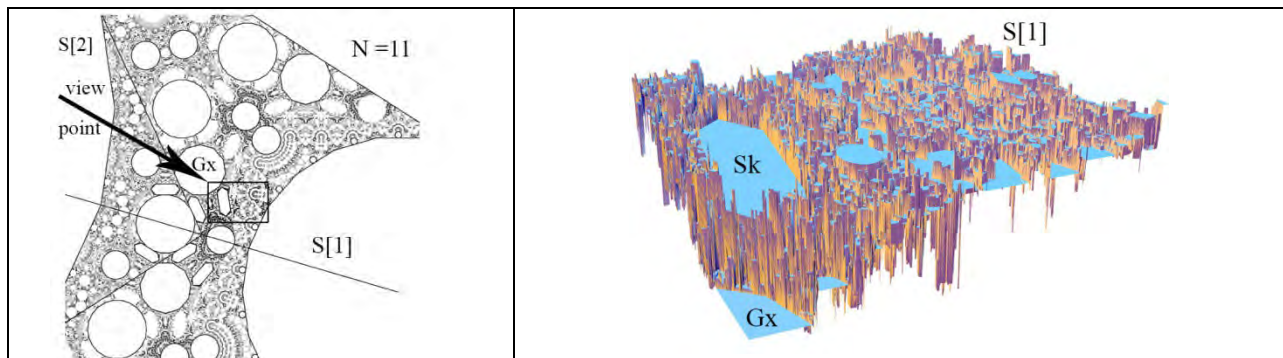
To analyze orbits like these, we use a tool borrowed from classical analysis. The winding number (rotation number) of an orbit is a measure of the average rotation per iteration

Definition 4.3 (winding numbers) For any N-gon, suppose a point p has step sequence $S = s_1, s_2, \dots$. The winding number of S is defined to be $\omega(S) = \frac{1}{N} [\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=1}^m s_j]$.

If S is a periodic step sequence with period k then $\omega(S) = \frac{1}{N} [\frac{1}{k} \sum_{j=1}^k s_j]$ so in this case the winding number is $1/N$ times the mean number of steps in one period of the step sequence. For a regular N-gon Lemma 4.1 says every S[k] will have constant step sequence $\{k\}$ with winding number k/N and the conjecture above extends this to the local families of the S[k].

In general these winding numbers are irrational and very difficult to compute. A knowledge of these sequences would be a major step toward understanding the dynamics of τ for any N-gon – regular or not. The only non-trivial regular cases known are quadratic with $N = 5$ by S. Tabachnikov in [T] (1995) and $N = 8$ by R. Schwartz in [S2] (2006). See Appendix E of [H3].

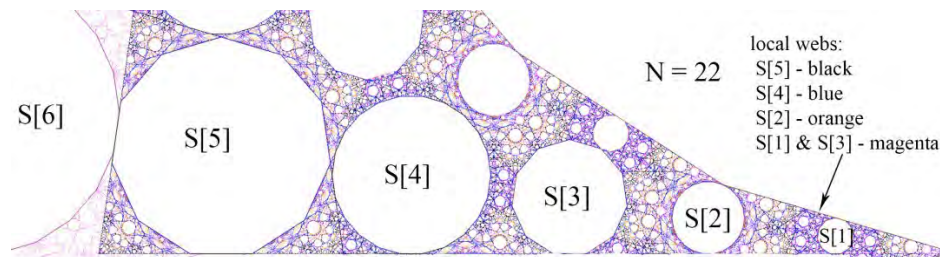
Example 4.7 On the right are winding numbers as relative heights in the vicinity of S[1] for $N = 11$. It is typical for larger ‘resonant’ tiles to have relatively small winding numbers. The S_k ‘skating rink’ has period 338 and $\omega = 41/169$ compared with S[1] at $1/11$ (which is minimal).



The Edge Geometry of Regular Polygons

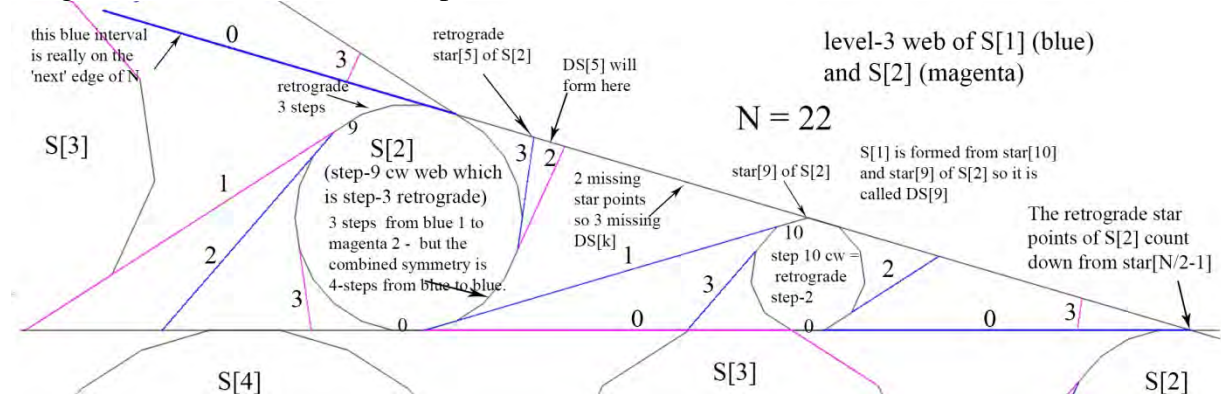
Our ultimate goal is to understand the limiting web W and the FFT is just a first step. The derivation of these local families in the GFFT was based on the step- k' web evolution of the $S[k]$ and it was only possible because these $S[k]$ 'parents' are closely related to each other and to N , so every $S[k]$ shares star points with its neighbors. This is no longer true for the 3rd generation and in general there is no clear path toward recursion and the small-scale geometry of the 3rd generation may be very complex. In H[6] and H[7] we will try to make some progress by focusing on the edge geometry of N . All regular N -gons have invariant regions where the geometry and dynamics of the 'edge' tiles $S[1]$ and $S[2]$ is shared with neighboring $S[k]$. For $N = 22$ below, this invariant region extends only to $S[5]$ because it shares an edge with $S[6]$.

Example 4.8 The invariant local geometry of $N = 22$ extends out to $S[5]$



The edge geometry of an N -gon is clearly dominated by the local webs of $S[1]$ and $S[2]$ but all the $S[k]$ of the invariant region would be expected to contribute to this geometry. For $N = 22$ above $S[2]$ has $k' = N/2 - k = 9$. That is an awkward step size, but all $S[k]$ are formed from two complementary cycles. For N even the complement cycle has $N/2 + 1 - (N/2 - k) = k + 1$ steps. This says that the 'retrograde' $S[2]$ web (which is just a right-side version of W) evolves in a step-3 fashion. By the same reasoning, $S[1]$ will have a step-2 retrograde web and it will be our convention to use an initial web interval that combines these two webs as shown below.

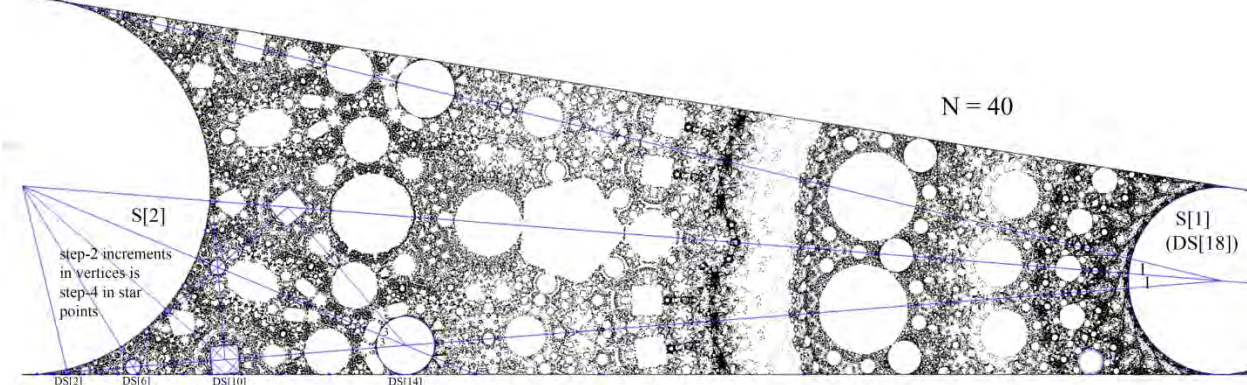
Example 4.9 The combined web evolution of $S[1]$ and $S[2]$ for $N = 22$ begins with an initial (magenta and blue) interval that spans both tiles. This will be our default initial interval 0.



As indicated above for N even, $S[1]$ and $S[2]$ will have step-2 and step-3 ccw webs, and by default we will combine their webs together as shown here. There will be a 1-step offset between these webs so the result is what we call a hybrid step-4 web for $S[2]$. This extra step is perfect for generating the shared next generation $S[k]$ tile of $S[2]$, which we call the $DS[k]$.

The early web evolution of these DS[k] is predictable because it is based on S[1] which is formed from star[N/2-1] and star[N/2-2] of S[2] – so it is called DS[N/2-2]. Because the combined web of S[1] and S[2] is step-4, the surviving DS[k] will count-down mod-4 to yield DS[5] and DS[1] for N = 22. This is the **Rule of 4 Conjecture** *For N even, counting down from S[1] at DS[N/2-2] the DS[k] will exist at least Mod-4.* Other ‘volunteer’ tiles may exist, but these DS[k] will evolve early and determine the symmetry of the web local to S[2]. This is illustrated with N = 40 below where the step-4 symmetry of S[2] is partitioned into step-2 increments yielding 4 star points between DS[k] (which here are simply S[k] of S[2]).

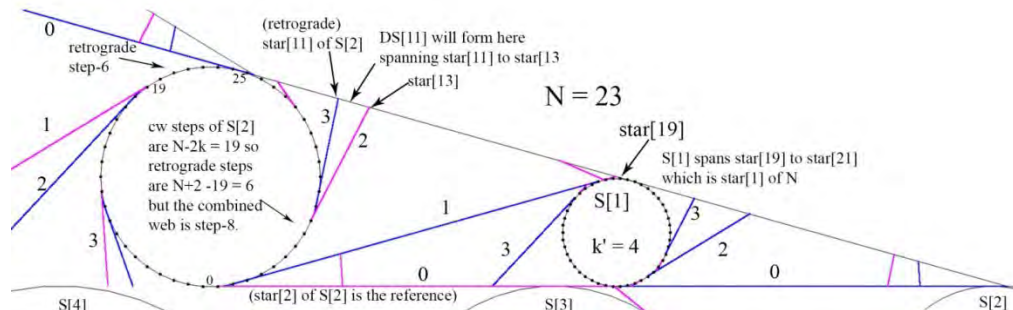
Example 4.10 The web of N = 40, showing the symmetry local to S[2]



The primary lines of symmetry are determined by S[1] which has $k' = N/2 - k = 18$ and retrograde step $k + 1 = 2$. At each effective star point of S[2], the web splits and DS[14] will have $k' = 6$. This determines the local symmetry of DS[14] as shown by the blue lines. DS[10] will have $k' = 10$ and be mutated into two squares since $N/\text{gcd}(10, N) = 4$. This implies that for N-even, the critical DS[1] and DS[2] tiles (if they exist) will have webs with $k' = N/2 - 1$ and $N/2 - 2$ and $\text{Mod}[N, N/2 - 1]$ is always 2 while $\text{Mod}[N, N/2 - 2]$ is 4, so there two will have step-2 and step-4 limiting webs. This means that DS[2] could play the role of a D[2], but for N even this DS[2] is not predicted to survive except in the 8k family and usually there is no matching M[2]. But the 8k+2 family has a predicted DS[3] and this can support D[2]s as well as M[2]s.

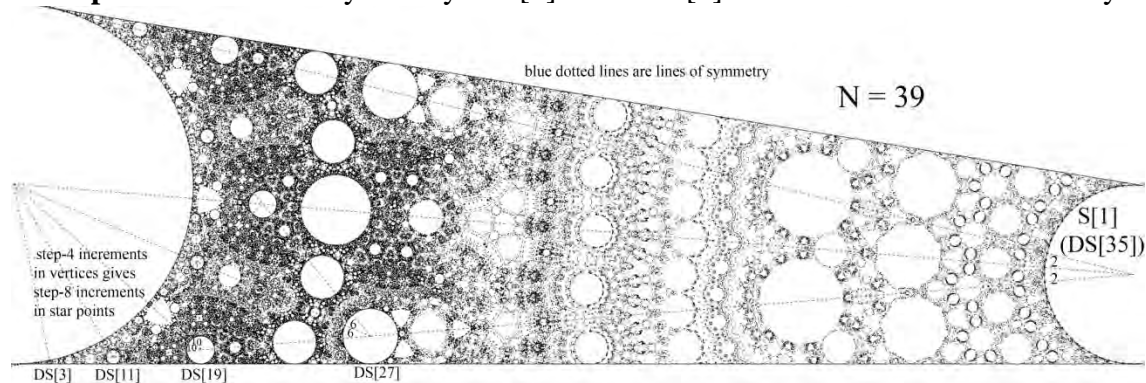
N-odd case is very similar but now S[1] and S[2] are 2N-gons, so the web steps are doubled from $N/2 - k$ to $N - 2k$ and the retrograde steps are also doubled at $k' = 2k + 2$. This means that S[1] will be retrograde step 4 and S[2] will be step-6, but the combined web of S[1] and S[2] will be step-8 as shown below. The DS[k] are now based on star[2] of S[2] and they will also be step-8 to match the effective star points. This is what we call the Rule of 8.

Example 4.11 The level-3 right-side webs of S[1] (blue) and S[2] (magenta) for N = 23



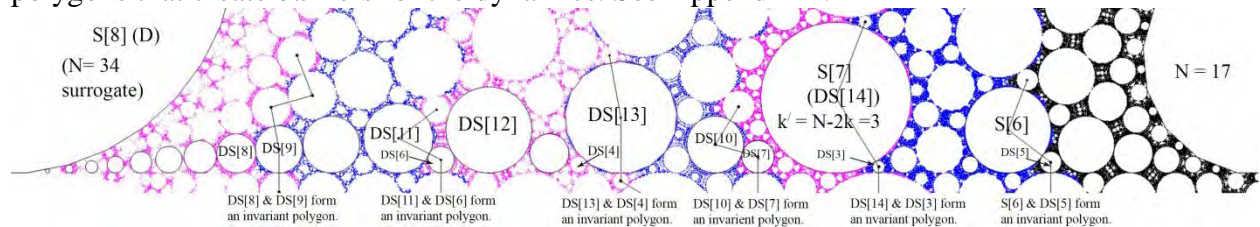
Rule of 8 Conjecture: For N odd, counting down from $S[1]$ at $DS[N-4]$ the $DS[k]$ will exist at least mod 8. The important caveat is that the $DS[k]$ tiles here are no longer part of the (ideal) First Family of $S[2]$ because they will be based at $star[2]$ of $S[2]$. They will be called the ‘star2 family’ of $S[2]$ and they will be D tiles relative to the (virtual) $S[k]$ of N . The $S[1]$ tile will always be $DS[N-4]$ so here it is $DS[35]$ and their combined $S[1]$ - $S[2]$ web, the effective star points of $S[2]$ will count down mod-8 from 35, to yield $DS[27]$, $DS[19]$, $DS[11]$ and $DS[3]$.

Example 4.12 The web symmetry of $S[2]$ local to $S[2]$ for $N = 39$ in the $8k + 7$ family



Since $S[1]$ and $S[2]$ are both $2N$ -gons the web steps of the $DS[k]$ will be $k' = 2(N/2) - k = N - k$ just like the even case. So $DS[3]$ and $DS[27]$ will be mutated with $k' = 36$ and 12 . Both have $N/\gcd(k', N) = 13$, so they will be the weave of 2 regular 13-gons. But unlike the N even-case these mutations will usually not be shared with $S[k]$ of N because when N is odd $k' = N - 2k$. For N odd the $S[k]$ tiles are $2N$ -gons and the GFFT will generate the step- k local families of the $S[k]$. This is done below for $N = 17$, but constructing the local $DS[k]$ of $S[2]$ is another issue. Here is a derivation using the Two-Star Lemma that should work for any $DS[k]$: $np = 2N$; $h = hS[2] * \tan[\pi/np] * \tan[k * \pi/np] / (\tan[\pi/np] / \tan[2 * \pi/np])$; $Mid = StarS2[[k]] + \{h * \tan[(np/2 - k) * \pi/np], 0\}$; $c = Mid + \{0, h\}$; $r = RadiusFromHeight[h, np]$; $DS[k] := RotateCorner[c + \{r, 0\}, np, c]$;

Example 4.13 The geometry of $N = 17$ showing 7 invariant regions defined by 6 ‘invariant’ polygons that create barriers for the dynamics. See Appendix II.



For N -odd, the First Family includes both even and odd cases so it contains two very different geometries at D and at N . The Twice-Odd Lemma implies that $N = 17$ and $N = 34$ share the same web, but that does not imply shared dynamics, so the question is what can we conclude about the dynamics and geometry of $N = 17$, based on the more predictable case of $N = 34$. The $8k+2$ Conjecture below will show that we have a good understanding of the local (magenta) dynamics of the $S[2]$ tile of D , but this magenta region (and all other regions) are invariant under τ , so no point in one region has an orbit that visits the other regions. This makes it difficult to relate the $N = 34$ dynamics with the $N = 17$ case. (Note that we used the GFFT here to construct the local step- k families of the $S[k]$ and help understand origin of the invariant polygons.)

The 8k+2 Conjecture

Suppose N is a regular polygon in the 8k+2 family. Since N is even, it is interchangeable with D and Definition 3.3 describes a well-defined (ideal) sequence of M[k] and D[k] tiles converging to star[1] of N which is also known as GenStar[N] or GenStar[N/2]. By reflective symmetry there will be an equivalent sequence converging to star[1] of S[2] which is the DS[2] of N. It is our convention to study the sequence there because S[2] can serve as a ‘surrogate’ N or D.

In this sequence N itself can be regarded as D[0] and the matching M[0] will be the S[N/2-2] penultimate tile of N. M[1] and D[1] will be the DS[1] and DS[2] of N which are simply the S[1] and S[2] of N since N is even. Then in the ideal sequence of Definition 3.3, for any positive integer k, $hM[k+1]/hM[k] = hD[k+1]/hD[k] = \text{GenScale}[N/2]$ since N is twice-odd.

(i) We conjecture that these M[k] and D[k] tiles will always exist and converge to GenStar[N] with geometric scaling $\text{GenScale}[N/2]$ and temporal scaling given by (ii) below

(ii) If D_k is the τ -period of D[k], it will satisfy the second-order difference equation

$$D_k = nD_{k-1} + (n+1)D_{k-2} \text{ where } n = N/2 \text{ and } D_1 = n \text{ and } D_2 = n^2$$

$$\text{The solution to this equation is } D_k = -\frac{n\left((-1)^k - (1+n)^k\right)}{2+n}.$$

Therefore the periods of the D[k] for $N = 2n$ will satisfy this closed-form equation.

$$\text{(iii) The ratio of the periods will be } D_k/D_{k-1} = \frac{(-1)^k - (1+n)^k}{(-1)^{1+k} - (1+n)^{-1+k}}$$

This sequence of ratios will clearly approach $n + 1 = N/2 + 1$ in the limit. It will begin with n and exceed n+1 on the second iteration and then alternate low-high relative to the limit.

(iv) Since ‘most’ M[k] form on the edges of the D[k-1] they will have the same limiting ratios and also satisfy the same basic difference equation as the D[k], but the new initial conditions will be $M_1 = \text{period of } M[1] = n$ and $M_2 = \text{period of } M[2] = n(3(n-1)/2 + 2)$, (This later period can be explained by noting that as shown below, ‘most’ D[k] arise in groups of 2 with 3 M[k] in each group plus the 2 M[k] of the single D[k]). The solution is

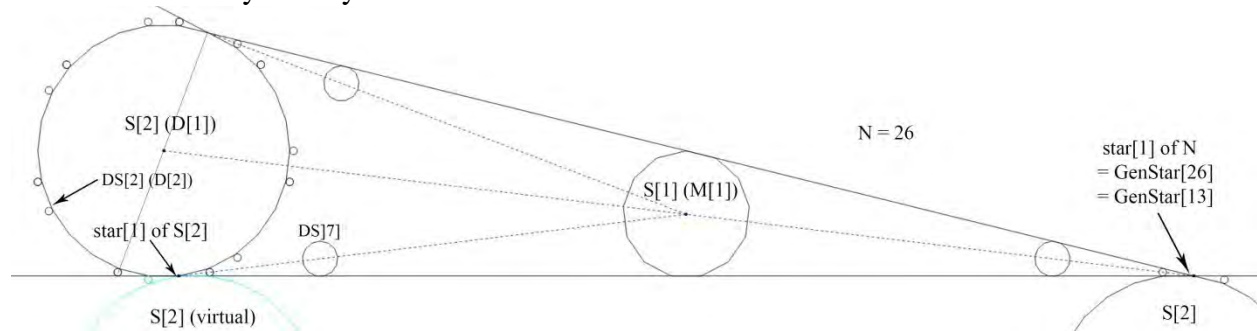
$$M_k = \frac{n\left((-1)^{1+k} + (-1)^k n + 3(1+n)^k\right)}{2(2+n)} \text{ (period of the } M[k] \text{ for } n = 2N)$$

(v) Each of the predicted DS[k] in the Edge Conjecture will also survive in each generation.

Note:Part (v) implies that the entire region local to S[2] should share the geometric scaling of the M[k] and D[k] tiles, but the remaining DS[k] may have very different temporal scaling. Working back from D[2] toward star[1] of S[2] we know both the geometric and temporal scaling so the local fractal dimension should be $\text{Log}[N/2+1]/\text{Log}[1/\text{GenScale}[N/2]]$ for $N = 8k+2$. This begins with 1.2411 for $N = 10$ and approaches $1/2$ with N. This does not imply that the subsequent generations will be self-similar and evidence says that typically they are not.

Example 4.14 $N = 26$ has algebraic complexity 6 and it is the first $8k+2$ case which is neither quadratic nor cubic in complexity. We will use it to help explain the difference equations for the evolution of the $D[k]$ and $M[k]$. The Edge Conjecture predicts that $S[1]$ will be $DS[11]$ and there will also be $DS[7]$ s and $DS[3]$ s as shown below. The solid blue center lines track the local web symmetry and this also applies to the First Family as shown by the $S[2]$ line of symmetry.

The geometry of $N = 26$ showing clusters of $D[2]$ tiles anchored by $S[3]$ s (not shown). The blue lines are lines of symmetry



Because the combined web of $S[1]$ and $S[2]$ is step-4 there will be k (from $8k+2$) of these $S[3]$ 'clusters' on either side of this line of symmetry. Each cluster has 2 $D[2]$ s and there is a lone $D[2]$ on the line of symmetry for a total of $4k+1$ $D[2]$ s. This is the ' $n = N/2$ ' in the difference equation. Since the period of any $S[k]$ is $N/\gcd(k,N)$, $S[2]$ has period n also. Therefore the period of the $D[2]$ s is n^2 . Every member of the $8k+2$ family will have $D[1]$ at period n and $D[2]$ at period n^2 . These are the initial conditions of the $p[k]$ equation. (Note that the $S[2]$ local to N is in a different group of $S[2]$ s so its small $D[2]$ is not counted here. However when $D[1]$ is replaced with $D[2]$ to count the next generation $D[3]$ s, these 'outliers' must be counted.)

At $D[1]$ there will be 13 magenta $D[3]$ around each $D[2]$ - each with their outliers. So there will be 14 outlier $D[3]$ s which are just visible here as magenta dots.

<p>Each $D[2]$ has 13 $D[3]$ just like $D[1]$ but there are also 14 'outliers' so the total count of the $D[3]$ is $13 \cdot 13^2 + 14 \cdot 13$</p> <p>(3rd generation $D[2]$ are in blue and 4th generation $D[3]$ are in magenta)</p>	<p>The periods of the first three $D[k]$ are: $D[1]$ at 13, $D[2]$ at 13^2 and $D[3]$ at $13^3 + 14 \cdot 13$ so the proposed difference equation for the periods D_k of the $D[k]$ is</p> $D_k = nD_{k-1} + (n+1)D_{k-2}$ <p>where $n = N/2$ and $D_1 = n$ and $D_2 = n^2$ Mathematica's solution is:</p> $D[k] \rightarrow -\frac{n((-1)^k - (1+n)^k)}{2+n}$ <p>as in the $8k+2$ Conjecture</p>
---	--

Therefore: $D[n,k] = -\frac{n((-1)^k - (1+n)^k)}{2+n}$ gives the τ -period of each $D[k]$ for $N = 2n$

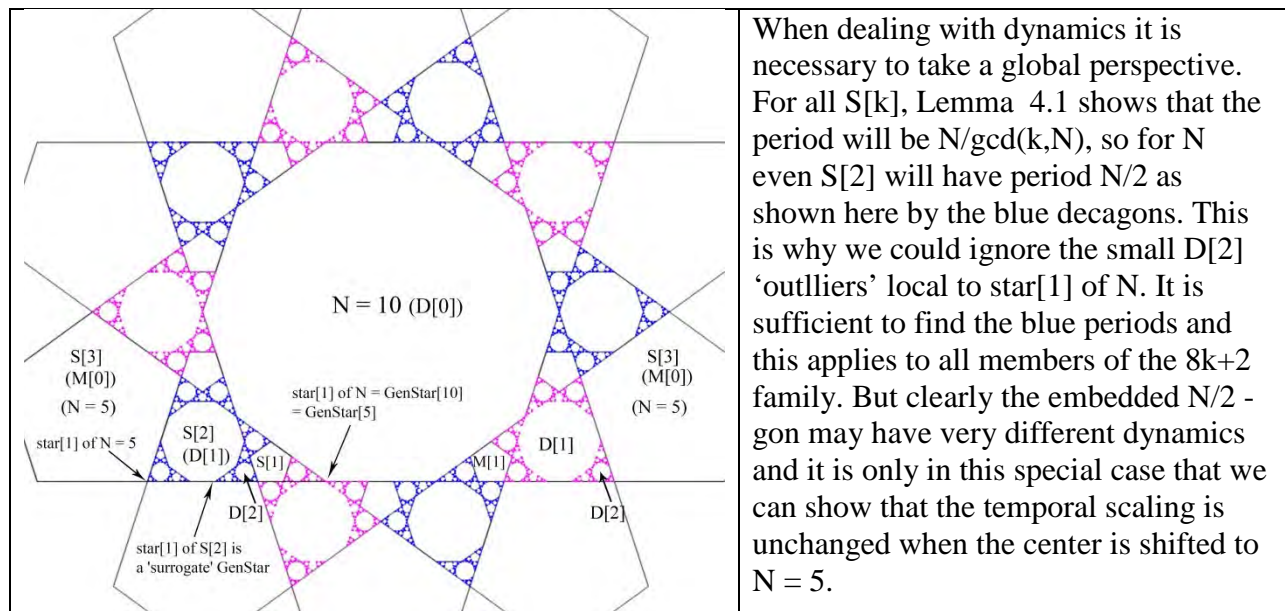
Table $[D[13,k], \{k,1,8\}] = \{13, 169, 2379, 33293, 466115, 6525597, 91358371, 1279017181\}$

The τ -periods of the $M[k]$ of $N = 26$ should satisfy $M[n,k] = \frac{n((-1)^{1+k} + (-1)^k n + 3(1+n)^k)}{2(2+n)}$

Table $[M[13,k], \{k,1,9\}] = \{ \{13, 260, 3562, 49946, 699166, 9788402, 137037550, 1918525778\}$

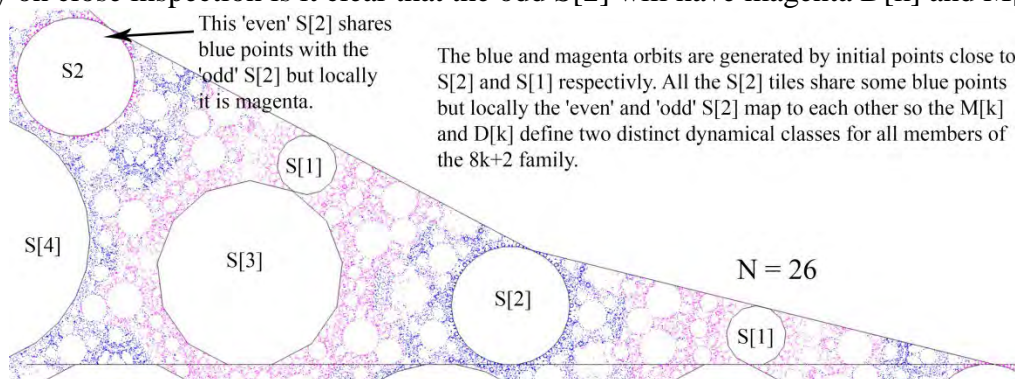
Because ‘most’ $D[k]$ come in clusters of 2 the $D[k]/M[k]$ ratio will approach $2/3$ in the limit. Since the centers of these tiles are known, we have verified the early cases. See $N = 34$ in [H7].

Example 4.16 $N = 10$ is the charter member of the $8k+2$ family. Since it has quadratic complexity there is just one non-trivial scale which it shares with $N = 5$, namely $\text{GenScale}[5] = (\sqrt{5} + 1)/2$. In Section 5 and [H6] we will look at the shared web of $N = 10$ and $N = 5$ and explain why both of these must have temporal scaling 6. Here we use $N = 10$ to illustrate the fact that even N -gons will have $S[2]$ orbits which ‘decompose’ into two congruent invariant regions.



This invariance occurs at all scales and these regions are typically ‘nested’ as shown below.

Example 4.17 For $N = 26$, the 13 ‘even’ and ‘odd’ $S[2]$ still map to other but only locally. The blue and magenta regions below are the orbits of points ‘close’ to $S[2]$ and $S[1]$ respectively. Only on close inspection is it clear that the odd $S[2]$ will have magenta $D[k]$ and $M[k]$.



Section 5. Examples of Singularity Sets

These examples will include the twice-odd pairs 5 and 10, 7 and 14, 11 and 22 and 13 and 26 as well as the twice-even cases $N = 12, 16$ and 20 .

For $N = 10$ the $8k+2$ Conjecture makes a number of assertions about the web and dynamics local to $S[1]$ and $S[2]$. The Edge Conjecture makes no $DS[k]$ predictions beyond $S[1]$ at $DS[3]$. Since $N = 10$ has quadratic complexity, the web it shares with $N = 5$ has just one non-trivial scale and should be fractal in nature. As indicated earlier, $N = 5$ and $N = 8$ are the only non-trivial regular cases where the dynamics and singularity sets have been studied in detail. In [T] (1995) S. Tabachnikov derived the fractal dimension of W for $N = 5$ using ‘normalization’ methods and symbolic dynamics and in [S2] (2006) R. Schwartz used similar methods for $N = 8$. In [BC] (2011) Bedaride and Cassaigne reproduced Tabachnikov’s results in the context of ‘language’ analysis and showed that $N = 5$ and $N = 10$ had equivalent sequences. (The Twice-Odd Lemma implies equivalent webs but not equivalent dynamics.) In [H3] we gave an independent analysis of the temporal scaling based on difference equations and that is reproduced in [H6]. Here we will give the implications of the $8k+2$ Conjecture.

Example 5.1 The temporal scaling of any $8k+2$ N -gon (with $N = 5$ as the lone $N/2$ case)

	<p>(i) The $D[k]$ of N will have period D_k where $D_k = nD_{k-1} + (n+1)D_{k-2}$ where $n = N/2$ and $D_1 = n$ and $D_2 = n^2$</p> <p>(ii) The $M[k]$ of N will have period M_k where $M_k = nM_{k-1} + (n+1)M_{k-2}$ where $n = N/2$ and $M_1 = n$ and $M_2 = n(3(n-1)/2 + 2)$</p> <p>(iii) The $D[k]$ of $N = 5$ will have period D_k where $D_k = nD_{k-1} + (n+1)D_{k-2}$ where $n = N$ and $D_1 = n$ and $D_2 = n(n+2)$</p> <p>(iv) The $M[k]$ of $N = 5$ will have period M_k where $M_k = nM_{k-1} + (n+1)M_{k-2}$ where $n = N$ and $M_1 = 2n$ and $M_2 = 2n^2$ (these will be $2D_k$ for $N = 10$)</p>
--	--

The solutions are:

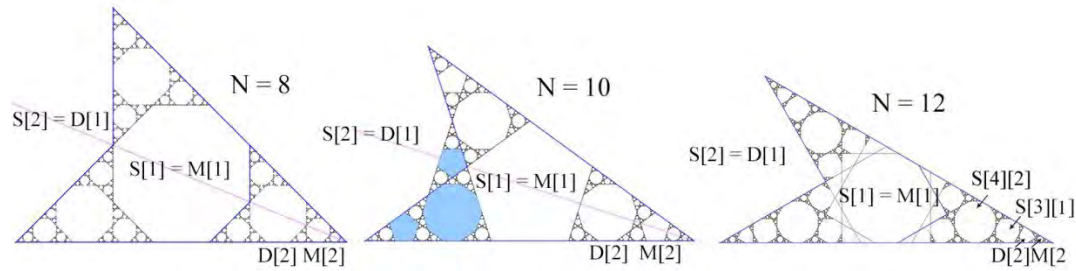
$(i) n = N/2: D[n, k] = -\frac{n((-1)^k - (1+n)^k)}{2+n}$	$(ii) n = N/2, M[n, k] = -\frac{n((-1)^{1+k} + (-1)^k n + 3(1+n)^k)}{2(2+n)}$
---	---

The first few periods are: $D_k: \{5, 25, 155, 925, 5555, 33325, 199955, 1199725, 7198355\}$ and $M_k \{5, 40, 230, 1390, 8330, 49990, 299930, 1799590, 10797530\}$

$(iii) n = 5, D[n, k] = \frac{n((-1)^k + (-1)^k n + 3(1+n)^k + n(1+n)^k)}{(1+n)(2+n)}$	$(iv) n = 5, M[n, k] = -\frac{2n((-1)^k - (n+1)^k)}{2+n}$
--	---

The first few periods are: $D_k \{5, 35, 205, 1235, 7405, 44435, 266605, 1599635, 9597805\}$ and $M_k: \{10, 50, 310, 1850, 11110, 66650, 399910, 2399450, 14396710\} = 2D_k$ for $N = 10$.

Example 5.2 Comparison of the edge geometry of the quadratic polygons $N = 8, 10$ and 12



These webs have just one non-trivial primitive geometric scale which is $\text{GenScale}[N]$ (or $\text{GenScale}[N/2]$ for $N = 10$). This is $hD[k]/hD[k-1] = hM[k]/hM[k-1]$, so it is also the scale of the darts (or triangles) which are anchored by $M[k]$ or $D[k]$.

In each case the magenta ‘renormalization’ line shows how the initial dart (or triangle) is mapped to a self-similar version of itself under τ^k for some k . As expected the $N = 8$ and $N = 12$ cases are closely related since their cyclotomic fields are generated by $\{\sqrt{2}, i\}$ and $\{\sqrt{3}, i\}$.

(i) For $N = 8$ (&12), the $D[k]$ and $M[k]$ are identical except for size. There are 3 darts in this invariant region and they are anchored by $D[k]$ s and each $D[k]$ is surrounded by 3 $M[k]$ s - so the $M[k]$ have temporal scaling of 9.

(ii) The temporal scaling of $N = 10$ is 6 as predicted by the $8k+2$ conjecture and Example 5.1. Note that the ‘next-generation’ light-blue region shown here is composed of two overlapping ‘towers’ containing an $M[k]$ and each $M[k]$ is surrounded by 3 $D[k+1]$ s for a temporal scaling of 6 for the $D[k]$. Since these towers form a sequence converging to $\text{star}[1]$ of $M[1]$ this helps to explain why the $D[k]$ s overall should have this same scaling. This is a non-trivial fact and as explained earlier, the key issue is the relationship between the decagons and pentagons.

(iii) For $N = 12$, each dart is anchored by an $S[4]$ and each $S[4]$ is surrounded by 3 $S[3]$ s for a combined scaling of 9, and in the limit each $S[3]$ will account for 3 $M[k]$ s, so the $M[k]$ scale by 27. This case is also not trivial and it is covered in more detail in Example 5.6.

Therefore the similarity (box-counting) dimension of the three webs should be:

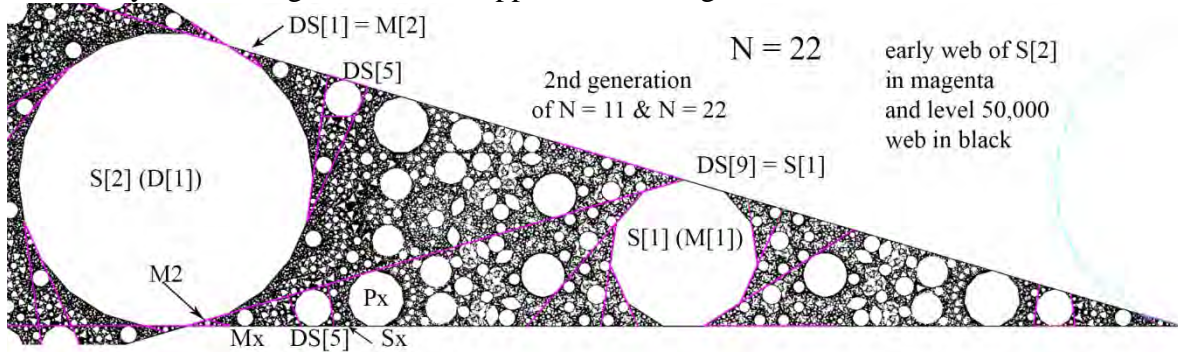
(i) $N = 5$ & 10: $\text{Log}[6]/\text{Log}[1/\text{GenScale}[5]] \approx 1.2411$ where $\text{GenScale}[5] = \text{Tan}[\pi/5]\text{Tan}[\pi/10]$

(ii) $N = 8$: $\text{Log}[9]/\text{Log}[1/\text{GenScale}[8]] \approx 1.2465$ where $\text{GenScale}[8] = \text{Tan}[\pi/8]^2$

(iii) $N = 12$: $\text{Log}[27]/\text{Log}[1/\text{GenScale}[12]] \approx 1.2513$ where $\text{GenScale}[12] = \text{Tan}[\pi/12]^2$

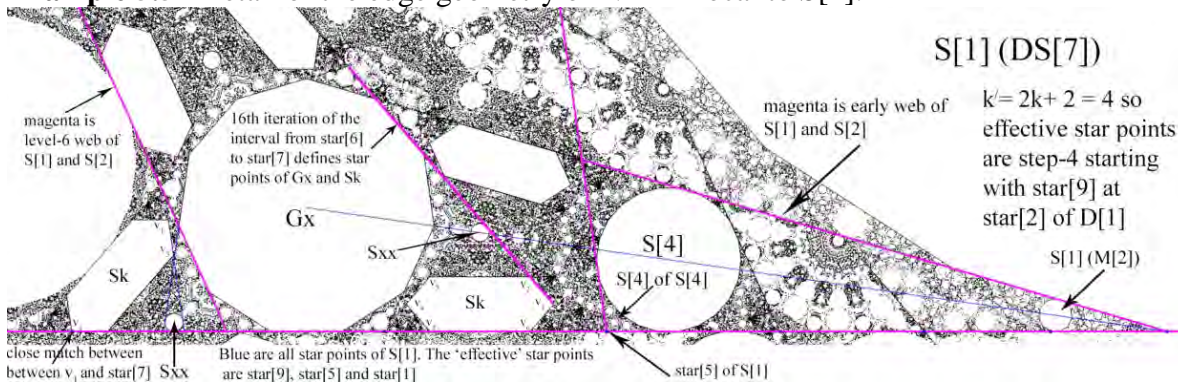
For compact self-similar sets such as these, the similarity dimension will match the traditional Hausdorff fractal dimension. It is no surprise that these dimensions are increasing, but this applies only to the quadratic family. For the cubic family and beyond, the webs are probably multi-fractal with a spectrum of dimensions. However it is likely that the maximal Hausdorff dimension will increase with the algebraic complexity of N , with limiting value of 2. See[LKV].

Example 5.4 ($N = 11$ & $N = 22$) These are the only ‘quintic’ cases. The two-elephant case from Example 1.2 takes place in the 2nd generation so these tiles exist on the edges of D which is a reflection of $N = 22$. This is an $8k+6$ family like $N = 14$ so there is a $DS[1]$ serving as an $M[2]$, but apparently it is not capable of generating $D[2]$ without the help of a $DS[3]$ and it appears that the weakly conforming Mx tiles that appear in the 2nd generation are remnants of failed $DS[3]$ s.



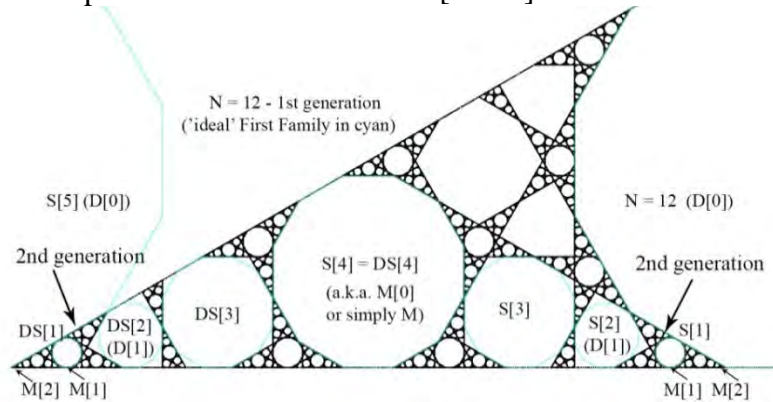
The web plot shows the predicted survival of $M[2]$ and $DS[5]$ along with $S[1]$ at $DS[9]$. This $DS[5]$ played a role in the earlier construction of Sx and the other ‘elephant’ was Px which is only weakly conforming to $D[1]$. In $H[6]$ we derive the parameters of Px along with Mx and Sx . The difficult cases were Mx and Px and it was only possible to derive their parameters by doing exact calculations with the web of $S[2]$. (These calculations for Mx are repeated in the Appendix to show that the toral digital-filter map and the complex-valued dual-center map give the same parameters. We will sometimes use the dual-center map in these plots because it is very efficient but the Dc map requires a change of origin, so in the plot above, $star[1]$ of N is at the origin.)

Example 5.5 Detail of the edge geometry of $N = 11$ local to $S[1]$.

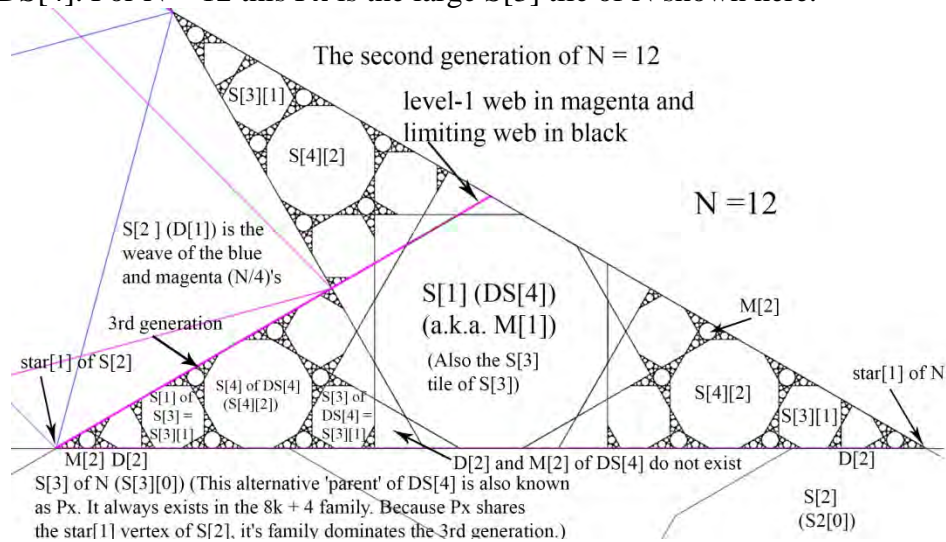


This $8k+3$ family has no canonical $DS[k]$ adjacent to $S[2]$, but relative to $S[1]$, the effective star points are mod-4 counting down from $star[N-2]$ at $star[2]$ of $S[2]$. Therefore the $8k+3$ and $8k+7$ families will have $star[1]$ effective and for $N = 11$ this provides support for an $S[4]$ as well as an $S[1]$. The Gx tile is a weakly conforming volunteer that also forms early in the web and is no doubt responsible for the Sk (See Example 4.7). In $[H6]$ we find the parameters of Gx and the Sk . Using Gx and one of the Sk as ‘elephants’ it is easy to find the parameters of the small Sxx tile, as shown by the blue extended edges above. This is the first documented ‘offspring’ of two tiles which are themselves volunteers so the (location) parameters of Sxx are very complex - but $hSxx$ is simple because Sxx is congruent to $S[4]$ of $S[4]$ shown in magenta. Because $S[1]$ itself is congruent to $D[1]$ of $N = 22$, there may be subtle connections between Gx , Sk , Sxx and the $N = 22$ volunteers Mx , Px and Sx .

Example 5.6 ($N = 12$). $N = 12$ is a quadratic polygon and the first non-trivial member of the $8k+4$ family. This is a very interesting family geometrically and algebraically. Since the mutation criteria for $S[k]$ in the twice-even case is $\gcd(N/2-k, N) > 2$, this family is the only one where $S[2]$ is mutated. This is a simple star[1] to opposite-side star[3] mutation, so $S[2]$ will always be the equilateral Riffle or ‘weave’ of two $N/4$ -gons. This is compatible with the Edge Conjecture which predicts the survival of a $DS[4]$. For $N = 12$, $S[3]$ is also mutated, but as always the M tile escapes mutation because it is $S[N/2-2]$.



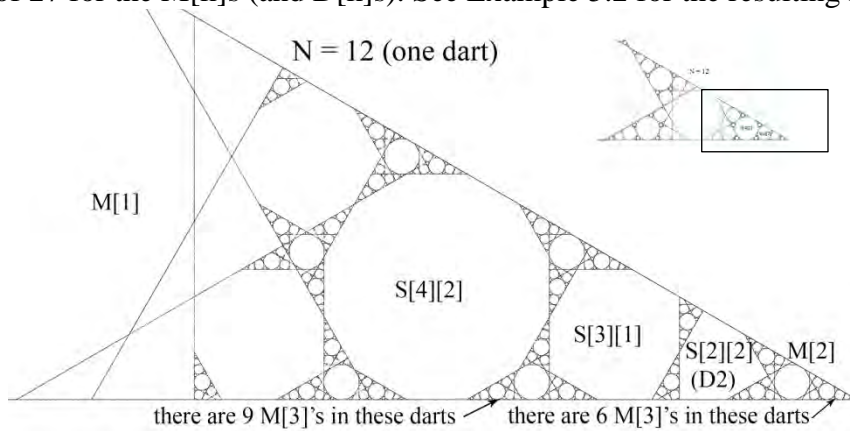
Below is the 2nd generation where the predicted $DS[4]$ is $S[1]$, also known as $M[1]$. The geometry local to $M[1]$ looks like an unfinished web but this is consistent with the fact that $S[2]$ is composed of two regular triangles which have trivial local webs. As N increases in the $8k + 4$ family, this local $N/4$ -web of $S[2]$ will be more complex, but apparently it will never yield ‘normal’ families for the $DS[4]$ s at the magenta extended vertices. However the blue vertices will be also be vertices of the underlying $S[2]$ and have a more promising geometry. It appears that for all members of this $8k+4$ family, the blue star[1] of $S[2]$ will be shared by an alternative ‘parent’ P_x of $DS[4]$. For $N = 12$ this P_x is the large $S[3]$ tile of N shown here.



The First Family of $S[3]$ does include $DS[4]$ as well as a next-generation $S[3][1]$ at the $S[1]$ position. This $S[3][1]$ survives the web along with $DS[4]$. (The $S[2]$ of $S[3]$ does not survive.) This $S[3][1]$ and symmetric copies then generate $M[2]$'s as their $S[3]$ tiles. We will show that in the limit each $S[3][k]$ will account for 3 $M[k+1]$ s, and since there will 3 of these $S[3][1]$ s in each ‘dart’ and there are 3 darts in each generation, the overall growth rate should be 27 as expected.

Here the count of $M[2]$ s is a little short with only 24 but that is because the mutation of $S[2]$ destroys potential $M[2]$ s (and $D[2]$ s) which would exist on the edges of $DS[4]$. We will show why these mutations will play a diminishing role with each new generation.

Below is an enlargement of one dart. The only $M[2]$ that is generated by a (mutated) $D[2]$ is at $star[1]$ of N . The remaining 5 $M[2]$ s are generated as step-3 tiles of the $S[3][1]$ so they have normal webs which will contain $M[3]$ s at the ‘next-generation’ step-1 positions. Therefore of the 23 ‘darts’ shown here only 3 are affected by the mutations - and this ratio will decrease with each new generation to yield a limiting count of 3 $M[k]$ s for each $S[3][k-1]$ and a limiting temporal scaling of 27 for the $M[k]$ s (and $D[k]$ s). See Example 5.2 for the resulting fractal dimension.

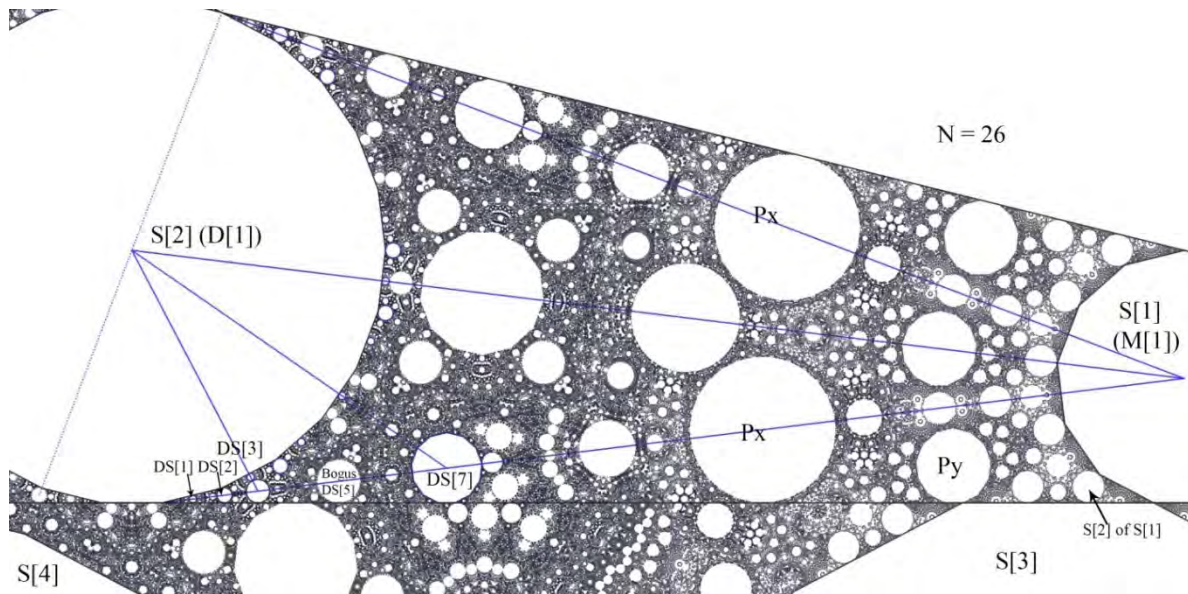


Another way to verify the temporal scaling for quadratic self-similar webs is to simply count the growth of the tiles using the τ -periods. Here the τ -periods of the $M[k]$ in the canonical invariant ‘star-region’ of Example 4.2 are 60, 942, 28292, 775356, 21055308,.. with ratios of 15, 30, 27.41, 27.15. These are the combined periods of the $M[k]$ at GenStar and their reflections about $S[4]$. Even though the local web has perfect reflective symmetry with respect to $S[4]$, the dynamics are different and this combined count helps to minimize these differences. The dynamics of any composite N -gon allows for the possible ‘decomposition’ of expected orbits unto groups of orbits with smaller periods. This makes it difficult to match tile counts with periods, but for self-similar webs, the effect of these exceptions diminishes with each generation and in the limit the τ -ratios will match the geometric ratios.

Returning to the First Family above, note that for N even, there is always an ‘extended edge’ relationship between N and M , with a variable number of intermediate ties. Here the intermediate tiles are $D[1]$ and $M[1]$, but with the mutation of $S[2]$ in the 2nd generation, these intermediate tiles become virtual tiles in the First Family of $DS[4]$.

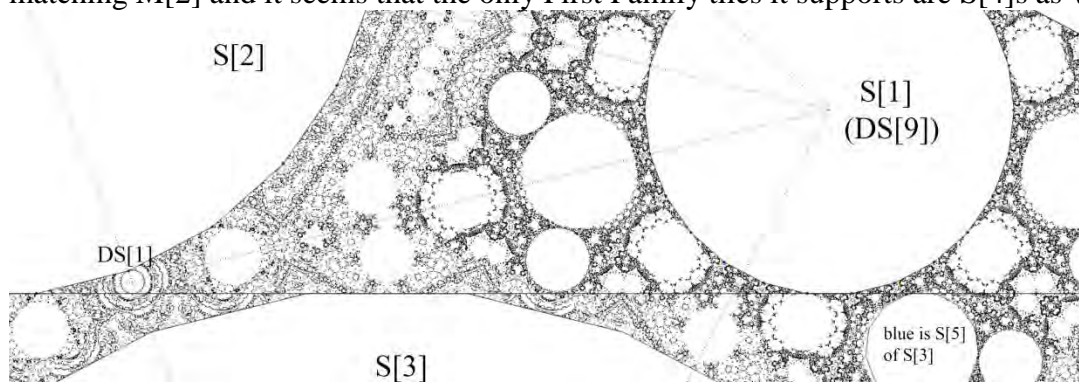
The **$8k+4$ Conjecture** states that these P_x ‘parents’ of $DS[4]$ always exist and come in two versions. $N = 12$ is the charter member of a mod-16 family $12+16j$ where P_x has $DS[4]$ at $S[3+8j]$. The next member is $N = 28$ where $DS[4]$ is $S[11]$ of P_x and this in turn defines P_x . The matching mod-16 family is based on $N = 20$ and here P_x is simply a D tile of $DS[4]$ but as evidenced by $N = 52$ the P_x may now be virtual. But these virtual embeddings are guaranteed by the FFT, so this branch may tell us little about the geometry at $star[1]$ of $S[2]$. Based on $N = 44$ in the $N = 12$ branch, even the real P_x may have First Families with no actual survivors except $DS[4]$ and the local geometry at $DS[4]$ is a disrupted because of the slight offset from $S[2]$.

Example 5.7 ($N = 26$ & $N = 13$) These two have algebraic order 6 along with $N = 21, 28, 36$ and 42. Since $N = 26$ is the first member of the $8k+2$ family with complexity that is neither quadratic or cubic, it was used as an example for the $8k+2$ Conjecture earlier. Below is what we call a symmetry plot where the blue center lines show local web symmetry. When N is even the combined $S[1]$ and $S[2]$ web will be step-4 and the Edge Conjecture predicts that there will be surviving mod-4 $DS[k]$. The salient feature of the $8k+2$ family is the early occurrence of $DS[3]$ s with their matching $DS[1]$ s and $DS[2]$ s. It appears that both of these are needed so that the $D[2]$ s will inherit the step-4 web structure of $S[2]$ and support sequences of $S[k]$ and $M[k]$ converging to star[1] of $S[2]$. The $8k+2$ Conjecture predicts that tiles like $DS[7]$ will also survive in subsequent generations but this does not imply that volunteers like P_x will survive. However for $N = 26$ the P_y appear to survive in the web local to $S[1]$ and their parameters are easy to find.



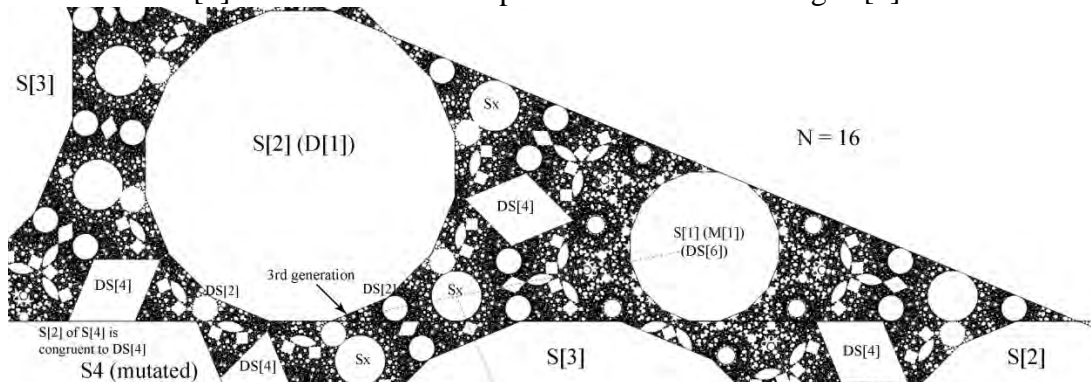
The Twice-Odd Lemma implies that the $S[11] (M)$ tile of $N = 26$ will be a surrogate $N = 13$. But this tile is a different invariant region so its dynamics would be expected to be very different and we have no theory that relates this $M[0]$ tile with $M[1]$ here.

Below is a symmetry plot for $N = 13$ in the $8k+5$ family. The Rule of 8 makes no predictions except for $S[1]$ at $DS[9]$ and an isolated $DS[1]$ which could serve as a $D[2]$, but there is no matching $M[2]$ and it seems that the only First Family tiles it supports are $S[4]$ s as vertex tiles.



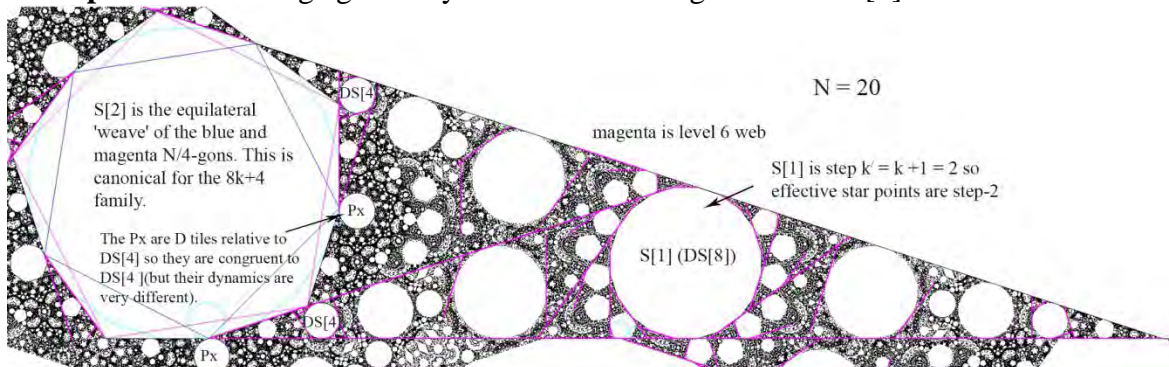
Example 5.8 - The edge geometry of $N = 16$.

$N = 16$ has ‘quartic’ complexity along with $N = 15, 20, 24$ and 30 . Both $N = 16$ and $N = 24$ are in the $8k$ family so the Rule of 4 implies an isolated $DS[2]$ will exist and here it acts as a $D[2]$ and supports a chain of $D[k]$ tiles with an almost-perfect chain of matching $M[k]$.



The $S[4]$ of N is mutated since $k' = N/2 - k = 4$, and $\gcd(4, N) = 4$. Therefore the mutated $S[4]$ will consist of the octagonal weave of two squares. Normally any $DS[4]$ of $S[2]$ would inherit this same mutation, but here it seems the interaction between $S[2]$ and $S[4]$ turn $DS[4]$ into a simple rhombus as shown here. This is an extended edge version of the mutation, which is not unique in the very ‘lazy’ web. The real surprise here is the chain of $D[k]$ and $M[k]$ which apparently exist even though $M[2]$ is missing. The periods of the first 9 $D[k]$ are: 8, $16 \cdot 2$, $8 \cdot 57$, $16 \cdot 154$, $8 \cdot 2609$, $16 \cdot 6898$, $8 \cdot 121863$, $16 \cdot 322826$, which gives even and odd ratios of about 5.3 and 8.8.

Example 5.9 - The edge geometry of $N = 20$ showing a mutated $S[2]$



$N = 20$ is the second non-trivial member of the $8k+4$ family. The Mutation Conjecture predicts that $S[2]$, $S[5]$ and $S[6]$ will be mutated and the $8k+4$ Conjecture above gives expected implications of the $S[2]$ mutation. The ‘odd’ vertices of $S[2]$ will always survive the mutation and the ‘even’ vertices will be extended outwards toward the predicted $DS[4]$ tiles. In the mod-16 subfamily anchored by $N = 12$ it appears that the $DS[4]$ s will be in the First Family of Px ‘parent’ tiles centered at odd vertices of $S[2]$, but for the $N = 20$ subfamily these Px tiles may be virtual beginning with $N = 52$. Here with $N = 20$ the Px are still quite real and there is a surviving $S[3]$ of Px with a promising web which could support future generations. But in general very little is known about the geometry at star[1] of $S[2]$. Even in the $N = 12$ subfamily where the parent Px tiles exist, a case like $N = 44$ has a Px where none of the First Family tiles survive the web, and there is no natural path towards recursion. A tile like $S[2]$ with competing regular constituents is an example of a ‘quasi-regular’ polygon in Appendix G of [H3]. The special case of two congruent regular constituents is called ‘semi-regular’ in [S3].

Appendix I Exact calculations using symbolic dynamics

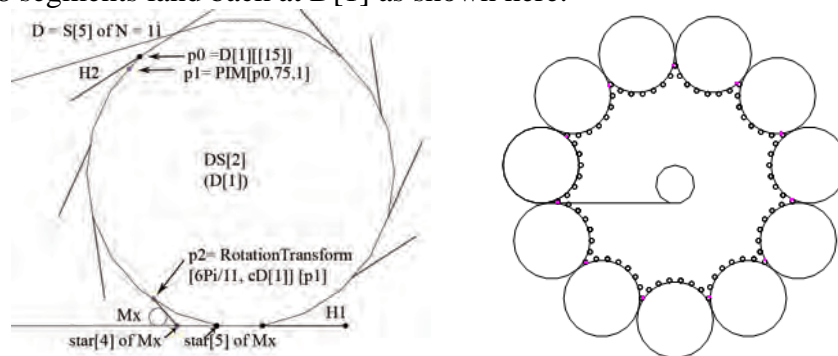
In [H2] (and in Example A3 below) we describe mappings from physics, astronomy, circuit theory and quantum mechanics which have singularity sets which may be locally conjugate to the outer-billiards map. Here we describe two such mappings and show how they can be used to perform exact calculations. The sample calculation will involve finding the parameters of the weakly conforming M_x tile of $N = 11$, as described in Example 5.4. We will first solve this problem with τ using ‘corner sequences’. These sequences are examples of ‘symbolic dynamics’ as first formulated by George Birkhoff and Stephen Smale. See [De].

The other two maps in question are the ‘digital-filter’ map of Chau & Lin [CL] and a refinement of a ‘dual-center’ map of Arek Goetz [Go]. Like the outer-billiards map, these are piecewise isometries based on rational rotations - also known as affine piecewise rotations. These three maps appear to have conjugate webs and this may be due to the fact that each can be reduced to a form of shear and rotation. Even though the webs are congruent, the dynamics appear to be very different but we shall show that they have a consistent form of symbolic dynamics.

Example A1 Use the evolution of the ‘web’ W to find the parameters of the M_x tile of $N = 11$ with (i) the outer-billiards map, (ii) the digital-filter map and (iii) a complex-valued Goetz map.

Part (i): The outer-billiards map. In Example 5.4, we noted that the M_x tile of $N = 11$ is a weakly conforming regular N -gon that occurs in the 2nd generation for $N = 11$ on the edges of D . Therefore $\text{star}[5]$ of M_x is $\text{star}[1]$ of $DS[1]$ as shown below. By the Two-Star Lemma, the parameters of M_x can be determined by finding another star point of M_x . In this example we will find the $\text{star}[4]$ point of M_x by tracing the web evolution of the interval $H1$ - in the context of N with radius 1. (By reflection, this evolution can equally be studied on the edges of $N = 22$.)

The interval $H1$ lies on the horizontal base edge of $N = 11$ so there are 11 such intervals equivalent to $H1$ under rotation, and the local τ - web determined by $H1$ includes the iteration of each of these rotated copies. Under τ , these 11 regions map to each other and after 99 iterations of each interval, 8 segments land back at $D[1]$ as shown here.



The interval $H2$ arises after just 13 iterations. Like all First Family vertices, p_0 is in \mathbb{Q}_{11} and we suspect that it maps to p_1 (as a one-sided limit) - and hence determines the offset of p_2 . These vertices technically have no image under τ , so we will find p_1 using the ‘surrogate’ orbit of a point that is close to p_0 and on the interval $H2$. (Typically intervals like $H2$ will get truncated under iteration, but by inspection it is clear that the inner portion of $H2$ survives well beyond the few hundred iterations needed here.)

Here are the calculations using Mathematica:

(i) Since the image under τ of any edge is a parallel edge, the slope of H2 is known, so set p0N to be a point on H2 within 8 decimal places of p0. **Orbit = NestList[τ , p0N, 200]** (an approximate orbit but initially reliable and any errors are easy to detect).

(ii) Since $\tau(p) = 2c_j - p$ for some vertex c_j of $N = 11$, $\tau^k(p) = (-1)^k p + 2Q$ where Q is a $+/-$ alternating sum of vertices. Every τ -orbit determines a sequence $\{c_k\}$ of vertices and the matching indices are sufficient to find Q and determine the orbit. The study of these partition sequences is called 'symbolic dynamics' so we will call them S-sequences. The Mathematica module IND will use τ to find the S sequences to any depth (once again with possible error).

$$S[p0N,150] = \mathbf{IND[p0N,150]} = \{11,5,10,4,9,3,8,1,6,11,5,10,4,9,2,7,1,6,11,5,\dots\}$$

Note that these indices initially advance by $\{5,5,5,5,5,4\} \pmod{11}$ because D is $S[5]$ with step sequence $\{5\}$ and $D[1]$ has (periodic) step sequence $\{5,5,5,5,5,4\}$. Here this sequence will eventually break down. In general no web point can have a periodic orbit because these points have no inverse. We will use these indices in pairs, using the 'return' map $\tau^2(p) = p + 2(c_k - c_j)$.

(iii) $P1 = \mathbf{PIM[p0N,75,1]}$ will take IND and these 150 indices in pairs and reconstruct the orbit, while $P3 = \mathbf{PIM[p0N,75,3]}$ will construct a step-3 version of this orbit, which is called a 'projection' or algebraic graph as defined in [S2]. To get an exact orbit, simply use p0 instead of p0N.

$$P1 = \mathbf{PIM[p0,75,1]} ; p1 = P1[[75]] = \tau^{150}(p0) ; p1[[1]] = -6\text{Cos}\left[\frac{3\pi}{22}\right] - \text{Cos}\left[\frac{\pi}{11}\right] \text{Cot}\left[\frac{\pi}{22}\right] + 4\text{Sin}\left[\frac{\pi}{11}\right] + 8\text{Sin}\left[\frac{2\pi}{11}\right] + \text{Cot}\left[\frac{\pi}{22}\right] \text{Sin}\left[\frac{\pi}{11}\right] \text{Tan}\left[\frac{\pi}{11}\right] - \text{Sec}\left[\frac{\pi}{22}\right] \text{Sin}\left[\frac{\pi}{11}\right] \text{Sin}\left[\frac{5\pi}{22}\right] \text{Tan}\left[\frac{\pi}{11}\right]$$

(iv) Rotate by $6\pi/11$ about the center of $D[1]$: $p2 = \mathbf{RotationTransform[6\cdot\text{Pi}/11, \mathbf{cD[1]}][p1]}$

(v) As indicated earlier, the slope of the web interval determined by $p2$ must match an edge of N . Here it has the same slope as the (right-side) star[4] edge of N , which we call slope4. $x1 = p2[[1]]$, $y1 = p2[[2]]$; $b = y1 - \text{slope4} \cdot x1$ so $\text{star}[4][[1]] = (1 - b)/\text{slope4}$

(vi) By the Two-Star Lemma $hMx = d/(\text{Tan}[5\text{Pi}/11] - \text{Tan}[4\text{Pi}/11])$ where d is the horizontal displacement of star[4] and star[5].

(vii) Of course the displacement d depends of hN but that dependence vanishes when hMx is divided by hN , which here is $\text{Cos}[\pi/11]$.

$$\mathbf{AlgebraicNumberPolynomial[ToNumberField[hMx/hN, GenScale[11], x]} = 1 - 23x - \frac{27x^2}{2} + \frac{x^4}{2} \text{ where } x = \text{GenScale}[11] = \text{Tan}[\pi/7] \cdot \text{Tan}[\pi/14].$$

Any other hN and matching hMx must yield this same ratio so this is a fundamental polynomial for Mx . Any 'canonical' tile with scaling in S_N will have such a polynomial. See Example A2.

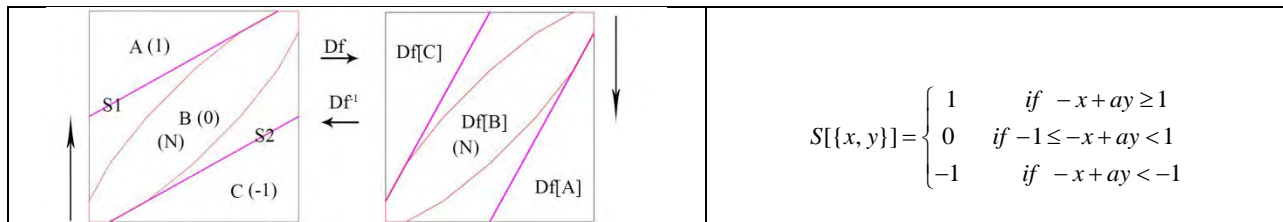
Part (ii) – The Digital Filter Map The Df map ([CL] and [H2]) is only compatible with the outer-billiards map when N is even, so it is necessary to work inside N = 22 (with hN = 1). This is not a burden and actually simplifies the calculations. Except for scale, the First Family is unchanged from part (i) - but S[9] is now playing the part of N = 11, so we will show that hMx/hS[9] satisfies the polynomial above.

The Digital Filter map Df: $[-1,1]^2 \rightarrow [-1,1]^2$ is defined as $Df[\{x,y\}] := \{y, f(-x + ay)\}$ where $f(v) = \text{Mod}[v+1,2]-1$ models a 2's complement sawtooth register. In matrix form (where $f(y) \equiv y$):

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = f \left(\begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \right) = \begin{bmatrix} y_k \\ f(-x_k + ay_k) \end{bmatrix}$$

Setting $a = 2\cos \theta$ the matching elliptical rotation $\begin{bmatrix} 0 & 1 \\ -1 & 2\cos \theta \end{bmatrix}$ is conjugate to a true rotation $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ but this conversion is optional.

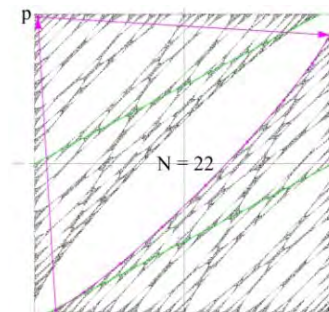
To generate the web of N = 22, set $\theta = 2\pi/22$ and here we are interested in generating symbolic orbits. The S sequences will be much simpler than τ , because the function f distinguishes just 3 regions - so Df is a piecewise isometry with three primary regions (atoms) which can be labeled 1 (overflow), 0 (in bounds), or -1 (underflow). The equations for these atoms are given here.



Under Df all points experience a rotation by θ and f determines the corresponding 'vertical' shear of -2, 0 and 2 respectively for A, B and C. The central B region is free of translation since it is 'in-bounds', so points will rotate by θ , and under iteration they will construct copies of N - one of which is shown here. The separators S1 and S2 define the maximal extent of this linear (elliptical) rotation, so they define the bounds of the three regions. Therefore if the S-sequence of a point p is known the Df map is simply $DfS[\{x,y\},k_] := \{y, -x+ay-2S[[k]]\}$

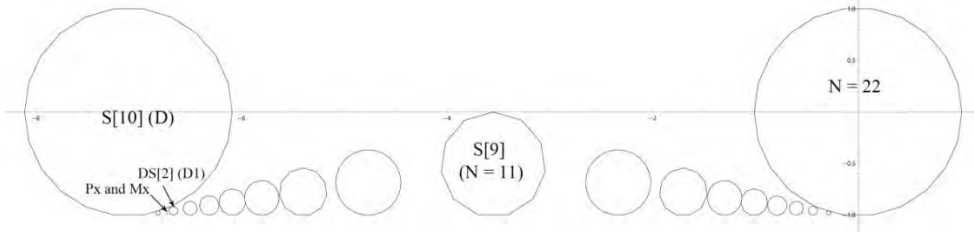
Example: The first 10 points in the Df orbit of cDS[1] for N = 22. Set $p = \text{TrToDf}[cDS[1]]$ where TrToDf is a change of coordinates between traditional Euclidean space and Df toral space.

Like all points adjacent to D, $p = \{x,y\}$ will be in an 'overflow' so $Df[p] = \{y, -x+ay-2\}$ as shown here. The point $\{y, -x+ay\}$ is 'out of bounds' at top right and the $\{0,-2\}$ shear is the f correction. Since $Df[p]$ is 'in-bounds', it allows for 8 consecutive central rotations, but the last rotation will yield an 'underflow' which will generate a compensating displacement back to an overflow position to repeat the cycle. Therefore the S sequence will be $\{1,0,0,0,0,0,0,0, -1,1\}$ and this periodic sequence determines the orbit of p.

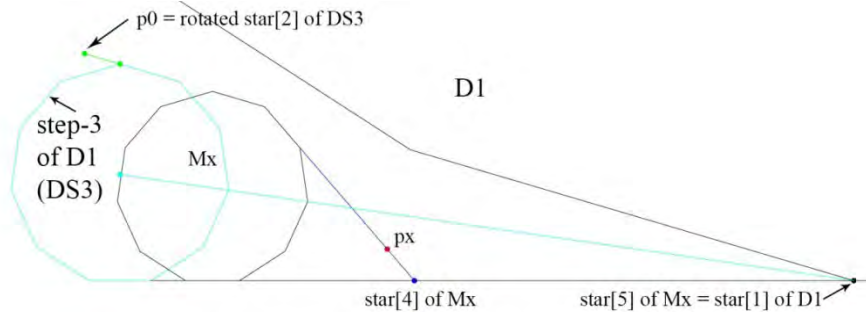


To get the level-k S sequence of any point p, first generate Orbit = NestList[Df,p,k] and then apply the function S to the elements of this orbit: Sequence = S/@Orbit.

As a guide we will use part (i) above to construct the expected Mx tile inside N = 22



The closest connection between Mx and the First Family of D1 appears to be the (virtual) step-3 tile of D1 – which we call DS[3][2] or simply DS3. We will use DS3 to define an interval that maps to Mx. The green interval shown below is a rectified portion of an edge in W, and under Df this interval will map to the blue interval – when rectified. Therefore p0 maps to px (using Df) and p0 is exact. It only takes 136 iterations to accomplish this – but an exact calculation will be awkward with Df - so we will use surrogate orbits and DfS instead.



Here are the calculations :

(i) Use $p_1 = \text{TrToDf}[p_0]$ to generate an approximate Df orbit of length 140 using $w = 2\text{Cos}[2\text{Pi}/22]$ to 30 decimal places. $\text{Orbit} = \text{NestList}[\text{Df}, p_1, 140]$; $S = S/@\text{Orbit} = \{1, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, \dots\}$ (As indicated earlier the points at the foot of D are typically in overflow positions so sequences such as this are common. Underflow must map to overflow because -1 and 1 are only one bit apart in 2 's complement.)

(ii) Generate an exact value of $\text{Df}^{137}[p_1]$ using DfS and exact $w = 2\text{Cos}[2\text{Pi}/22]$. Set $q = \text{Range}[140]$; $q[[1]] = p_1$; **For**[s=1, s<=140, s++, q[[s+1]] = **Simplify**[**Re**[**DfS**[q[[s]], s]]];

p_{xx} (px unrectified) = $q[[137]] \approx \{-6.679854552975, -0.9991754019926052\}$

$px = \text{Simplify}[\text{Re}[\text{DfToTr}[p_{xx}]]]$; $px[[1]] = \frac{(-422 + 419(-1)^{1/11} - 414(-1)^{2/11} + 419(-1)^{3/11} - 422(-1)^{4/11} + 421(-1)^{5/11} - 420(-1)^{6/11} + 424(-1)^{7/11} - 424(-1)^{8/11} + 420(-1)^{9/11} - 421(-1)^{10/11}) / ((1 + (-1)^{2/11})(-1 + (-1)^{1/11} + (-1)^{6/11})^2)}$

(iii) Set $x_1 = px[[1]]$; $y_1 = px[[2]]$; $b = y_1 - \text{slope4} \cdot x_1$ (where slope4 is the slope of right-side star[4] of N as in part (i)). Therefore $\text{star}[4] [[1]] = (1-b)/\text{slope4} =$

$$\frac{i(-13 - (-1)^{1/11} + 20(-1)^{2/11} + 14(-1)^{4/11} - 14(-1)^{5/11} - 20(-1)^{7/11} + (-1)^{8/11} + 13(-1)^{9/11})}{(1 + (-1)^{1/11})(1 + (-1)^{2/11})^4} - 2\text{Cot}\left[\frac{\pi}{11}\right]$$

(iv) As in part (i), $hMx = d/(\tan[5\pi/11] - \tan[4\pi/11])$ where d is the horizontal displacement of $star[4]$ and $star[5]$.

(v) This displacement d is relative to hN so the ratio hMx/hN is the scaling field S_{11} , but N is a 22-gon, so it makes more sense to use $hMx/hS[9]$ where $hS[9] = \tan[\pi/22]/\tan[\pi/11]$.

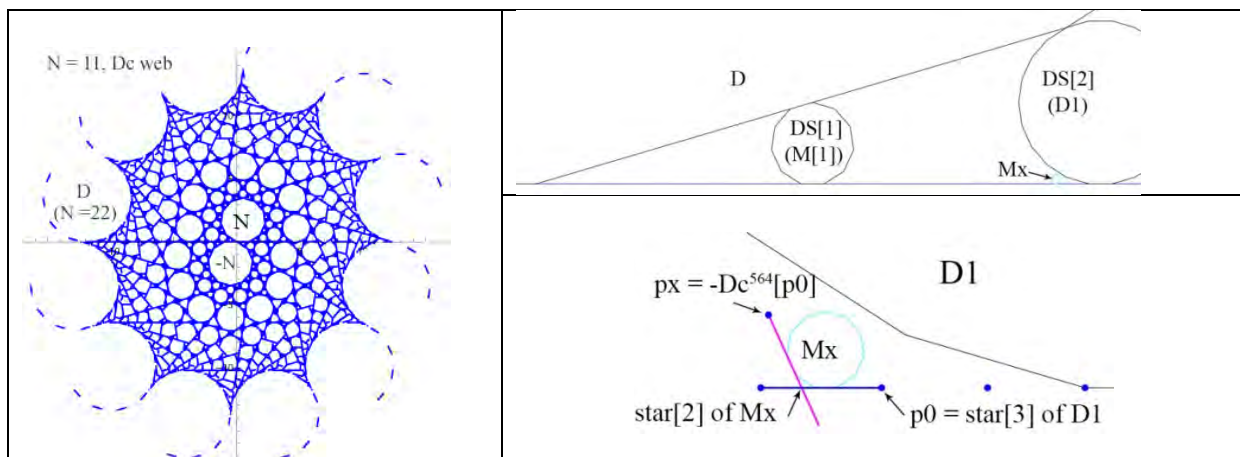
AlgebraicNumberPolynomial[ToNumberField[hMx/hS[9], GenScale[11]],x] gives

$$1 - 23x - \frac{27x^2}{2} + \frac{x^4}{2} \text{ as in part (i) above.}$$

Part (iii) A complex-valued Goetz Map

Perhaps the simplest map that reproduces the outer-billiards web for a regular N -gon is a y -axis version of a 'dual center' map of Arek Goetz. As described in [H2] this mapping has the form $Dc[z] = \text{Exp}[-Iw] \cdot (z - \text{Sign}[\text{Im}[z]])$ where $w = 2\pi/N$ and for a scalar v , $\text{Sign}[v] = 1, 0$ or -1 iff v is positive, 0 or negative. Therefore Dc is a pure (clockwise) rotation for points on the x -axis but in general it is a 'shear and rotate' where the shear has magnitude -1 above the x -axis and $+1$ below (in a manner similar to W with $sN = 1$). So if Dc is applied to the interval $[-1, 1]$ it will form a perfect N -gon above the x -axis and its negative below. If this initial interval is expanded as shown below for $N = 11$ (with $w = 2\pi/11$), then the edges of $-N$ will intersect the edges of N to simulate the interaction of the τ -domains. This juxtaposition actually occurs with W but not at the origin.

The web on the left below was generated by iterating the x -axis interval $[-12, 0]$, 200 times under Dc with $w = 2\pi/11$ (to 30 decimal places). In the limit, the region above (or below) the x -axis will be a perfect reproduction of the web for $N = 11$ with a side of 1 and $star[1]$ at the origin. It is an easy matter to scale the First Family (and Mx) to use as guides to track the web development at D . On the right we show a portion of the ideal second generation in black and the hypothetical Mx in cyan. Unlike the Df map, the webs tend to evolve in a predictable fashion from intervals on the x -axis and this makes it easy to find intervals that will map to Mx . The chosen interval is shown in blue in the enlargement on the lower right, and the magenta interval is the negative of the image of this blue interval under Dc^{564} .



Since the slope of this magenta line is known, all that is needed to find star[2] of Mx is some point like px on this line. Here are the calculations.

(i) The initial point is $p0 = \text{StarD1}[[3]] =$

$$\left(\frac{5}{2} - \text{Cot}\left[\frac{\pi}{22}\right]\text{Cot}\left[\frac{\pi}{11}\right] + \text{Tan}\left[\frac{\pi}{22}\right]\text{Tan}\left[\frac{\pi}{11}\right] + \frac{1}{2}\text{Cot}\left[\frac{\pi}{22}\right]\text{Cot}\left[\frac{\pi}{11}\right](1 - \text{Tan}\left[\frac{\pi}{22}\right]\text{Tan}\left[\frac{\pi}{11}\right](2 + \text{Tan}\left[\frac{\pi}{22}\right]\text{Tan}\left[\frac{\pi}{11}\right])) - \left(\frac{1}{2}\text{Cot}\left[\frac{\pi}{11}\right] + \frac{1}{2}\text{Cot}\left[\frac{\pi}{11}\right](-1 + \text{Tan}\left[\frac{\pi}{22}\right]\text{Tan}\left[\frac{\pi}{11}\right](2 + \text{Tan}\left[\frac{\pi}{22}\right]\text{Tan}\left[\frac{\pi}{11}\right]))\right)\text{Tan}\left[\frac{3\pi}{22}\right], 0\right)$$

(ii) Using $p0N = N[p0, 30]$ (30 decimal place approximation to p0), find the first 600 points in the (complex-valued) orbit: $\text{Orbit} = \text{NestList}[\text{Dc}, p0N, 600]$

(iii) $\mathbf{S[z_]:=Sign[Im[z]]}$; $\mathbf{Sx=S/@Orbit} = \{0, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, \dots\}$

(iv) Define a 'literal' version of Dc based on Sx, namely $\mathbf{DcS[z_, k_]:=Exp[CI \cdot w](z - Sx[[k]])}$ (with an exact w) to obtain the exact orbit of p0 using the Sx sequence from the surrogate orbit. Store the orbit in the sequence q:

q=Range[600];q[[1]]=p2[[1]]; For[s=1,s<=600, s++, q[[s+1]]=Simplify[DcS[q[[s]],s]]];

Note: By modifying F to allow approximate calculations of the Sign function, it is may be feasible to do calculations like this directly with NestList. One possible modified function is $\mathbf{F[z] = Simplify[Exp[-I*w](z - I*IntegerPart[Sign[N[Re[z]]])]}$. This tricks Mathematica into regarding the Sign output as exact. Normally Mathematica attempts to evaluate Sign in an exact fashion and these expressions get so complex that typically it fails after a few hundred iterations.

$$(v) \mathbf{q}[[564]] = \frac{6 - 8(-1)^{1/11} + 5(-1)^{2/11} - 2(-1)^{3/11} - (-1)^{4/11} + 3(-1)^{5/11} - (-1)^{6/11} + 5(-1)^{7/11} - 3(-1)^{8/11} + (-1)^{9/11} + 4(-1)^{10/11}}{-1 + 2(-1)^{1/11} - 3(-1)^{2/11} + 3(-1)^{3/11} - 2(-1)^{4/11} + (-1)^{5/11}}$$

The desired point is $px = -q[[564]] \approx -10.415046959467414 + 0.01643324914196498 \cdot I$

(vi) $x1 = \text{Re}[px]$; $y1 = \text{Im}[px]$; The slope of star[2] of Mx is $-\text{slope2}$ as defined using star[2] of N, so $b = y1 + \text{slope2} \cdot x1$ and $\text{star}[2] [[1]] = b/\text{slope2} \approx -10.407542146047456045$

(vii) Using the Two-Star Lemma with opposite sides $hMx = d/(\text{Tan}[2\text{Pi}/11] + \text{Tan}[5\text{Pi}/11])$ where $d = \text{star}[5] [[1]] - \text{star}[2] [[1]]$.

(viii) Since N has side 1, $hN = \frac{1}{2}\text{Cot}\left[\frac{\pi}{11}\right]$. Use this to convert hMx to a scale in S_{11} .

AlgebraicNumberPolynomial[ToNumberField[hMx/hN], GenScale[11], x]

$$= 1 - 23x - \frac{27x^2}{2} + \frac{x^4}{2} \text{ as in parts (i) and (ii)}$$

Since the star[3]-star[4] interval of D1 generates Mx, it can be regarded as a 'mutated' DS3[2]. Likewise Px is generated by the star[6] to star[7] interval so it may be a modified DS6[2].

The historical connection between these maps is illustrated in Example A3 below - which shows that Df is equivalent to a sawtooth (tuned) version of the classical Standard Map while Dc is a sawtooth (tuned) version of a kicked harmonic oscillator. The connection is that the Standard Map is equivalent to a kicked 'free' rotor in zero gravity while a harmonic oscillator is assumed to be affected by gravity - and hence has a natural frequency of oscillation.

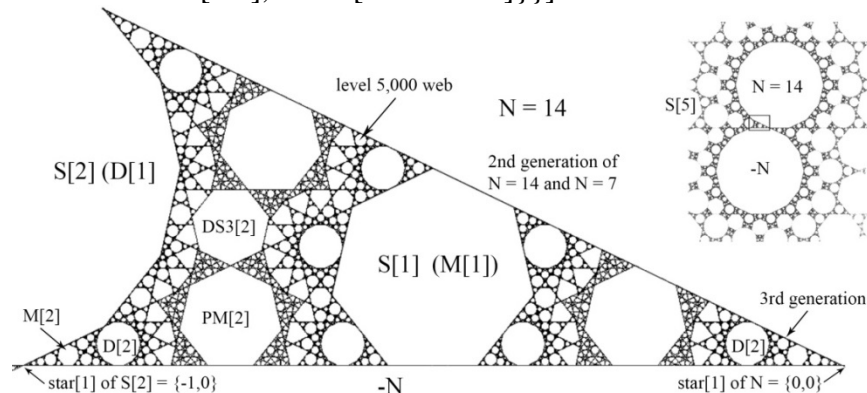
The classical Twist Map of J. Moser is also a ‘free rotor’ but it allows for the possibility of periodic kicks so the Standard Map illustrates the case of a Twist Map with these (ideal) periodic ‘gravitational perturbations’ turned on - and the issue is whether there will be a non-zero measure of initial conditions which yield ‘integrable’ solutions to the matching Hamiltonian.

It is still not clear what role gravity plays in quantum mechanics. In the late 50’s P.B. Harper and others used a kicked harmonic oscillator with a natural frequency of oscillation to model both classical and quantum diffusion - based on a stationary Schrodinger equation. This is illustrated in Example A3 for $\omega = 2\pi/4$ ($N = 4$) which shows initial resonances similar to the Standard Map – and then diffusive breakdown and weak mixing as K increases.

The Df and Dc maps are very efficient ways to generate the local web for a regular N-gon but Dc has a number of advantages and we will be using it as an aid toward publishing a detailed ‘4K’ catalog of edge geometry in [H6]. For this purpose the juxtaposition of N and $-N$ is ideal and the natural +/- symmetry of W is augmented by reflective symmetry to yield very efficient maps.

Below is an example from $N = 14$ where we iterate 1,000 points in the interval $H = \{-2, -1\}$ at a depth of 5,000. (The interval $\{-1, 1\}$ will generate N and $-N$ in a period N orbit.) Here we crop these 5 million web points and their negatives and reflections to the desired region. (Less than 1 minute to generate and 1 minute to crop on a modest computer.)

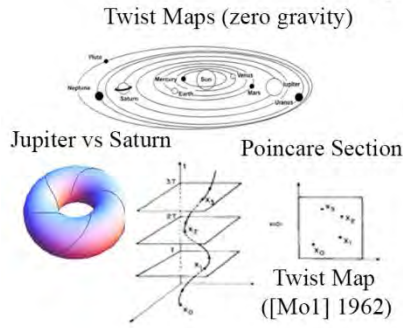
Example A2 (The edge geometry of $N = 14$) $Dc[z_]:=Exp[-I*w]*(z-Sign[Im[z]]);$
 $w=N[2*Pi/14,35];(35\text{ decimal places}); H=Table[x,\{x,-2,-1,.001\}]; Web =$
 $Table[NestList[Dc,H[[k]],5000],\{k,1,Length[H\}]; RealWeb = \{Re[\#],Im[\#]\}&/@Web;$
 $WebPoints = Crop[Union[RealWeb,-RealWeb,Reflection[RealWeb]]]; (about 450,000\text{ points})$
 $Graphics[\{AbsolutePointSize[1.0],Point[WebPoints]\}]$



Example 5.3 discussed the $N = 14$ and $N = 7$ families, and here we examine the scaling. The scaling fields S_7 and S_{14} are generated by $x = \text{GenScale}[7] = \text{Tan}[\pi/7] \cdot \text{Tan}[\pi/14]$. Inside $N = 14$, $S[5]$ is the surrogate $N = 7$, so by convention all heptagons are scaled relative to $S[5]$. $M[1]$ (a.k.a.M1) is not a true 2nd generation $N = 7$ (except by scale) because the edge dynamics are very different with PMs in place of $S[2]$ s. However $M[2]$ is a valid 3rd generation ‘matriarch’ with the same edge geometry as $N = 7$. The even and odd generations appear to be self-similar.

hM1/hS[5]	hD1/hN	hDS3[2]/hS[5]	hPM[2]/hS[5]	hS[5]/hN	hM2/hS[5]	hD2/hN
x	x	$\frac{1}{2} - 4x - \frac{3x^2}{2}$	$-\frac{3}{8} + \frac{17x}{4} + \frac{9x^2}{8}$	$\frac{3}{7} + \frac{3x}{7} - \frac{x^2}{7}$	x^2	x^2

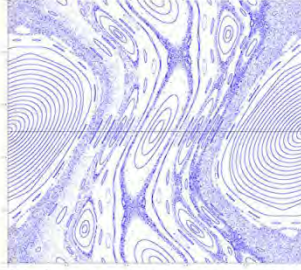
Example A3 - Some classical maps relevant to this paper



$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k + x_k \end{bmatrix}$$

(Here y is θ and x is momentum (constant))

Standard Map (Chirikov 1969) ($K = .971635406$)

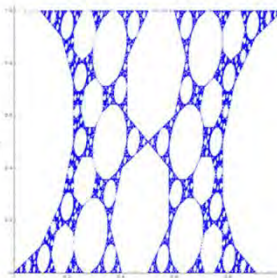


This is a 'Mod -1' periodically perturbed Twist Map showing resonant 'islands' of stability on all scales

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k + (K \sin 2\pi y_k) / 2\pi \\ y_k + x_k + (K \sin 2\pi y_k) / 2\pi \end{bmatrix}$$

(gravity turned on)

Sawtooth Standard Map ([A] 2001) ($k = 2\text{Cos}(2\text{Pi}/14) - 2$)



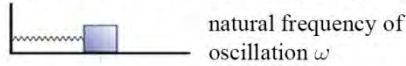
The $S[k]$ of $N = 14$ are the primary resonances of this Standard Map tuned to 'Ch14'

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k + k\text{Saw}(y_k) \\ y_k + x_k + k\text{Saw}(y_k) \end{bmatrix}$$

($\text{Saw}(x) = x - 1/2$ on $[0,1]$)

(The outer-billiards web W can be regarded as a sawtooth (tuned) version of the Standard Map)

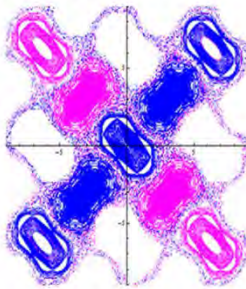
Harmonic Oscillator (non-zero gravity)



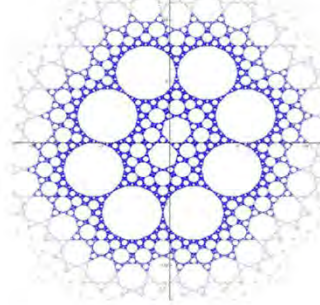
As in the Standard Map there may be periodic 'kicks' (and possible friction)

These are equivalent under a change of variables

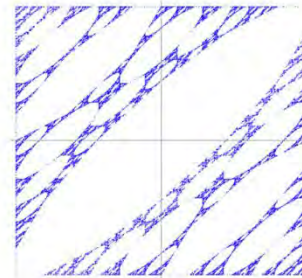
Kicked Harmonic Oscillator (KHO) Harper Map ($\omega = 2\text{Pi}/4$) ([Ha]1955)



Dissipative KHO (DKHO) ([H2] 2012) ($\omega = 2\text{Pi}/7$ ($N = 7$))



Digital Filter Map ([CL] 1988) ($a = 2\text{Cos}(\omega)$ - here $\omega = 2\text{Pi}/14$)



$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x_k + K \sin y_k \\ y_k \end{bmatrix}$$

(sometimes unequal kicks are applied to both x and y)

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x_k - \text{sgn } y_k \\ y_k \end{bmatrix}$$

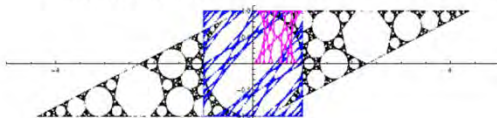
Replacing sine with sgn (sign) yields a sawtooth (tunable) map that we call the dual-center map. In complex form: $F[z] = \text{Exp}[-i\omega] * (z - \text{Sign}[\text{Im}[z]])$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = f \left(\begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \right) = \begin{bmatrix} y_k \\ f(-x_k + ay_k) \end{bmatrix}$$

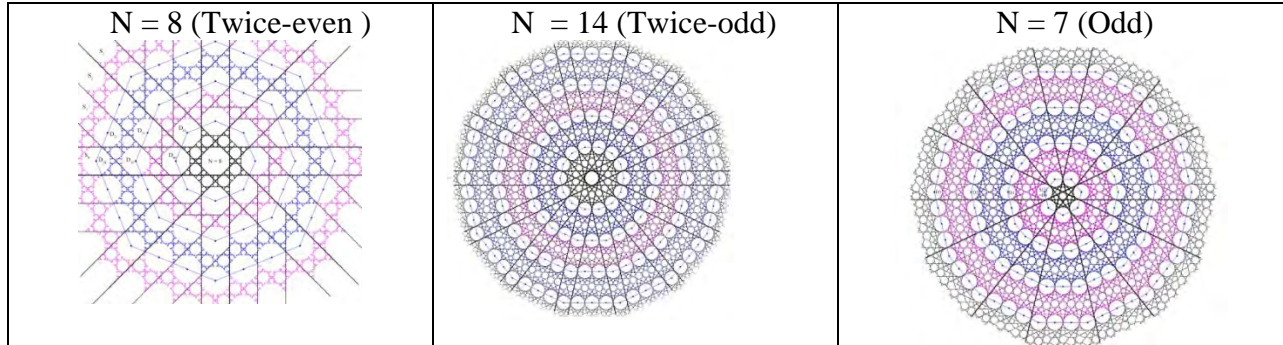
Since $a = 2\text{Cos}(\omega)$ this is conjugate to $\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$

The (sawtooth) register overflow function is $f(x) = \text{Mod}[x+1,2] - 1$ and $f(y) = y_k$ because y is always in range

Overlay of Sawtooth Standard Map (magenta), Digital Filter Map (blue) and rectified $N = 14$ web in black



Appendix II Large Scale Ring Structure of a Regular Polygon



(i) The concentric rings of D tiles first described in Shaidenko-Vivaldi [SV], and later in [H3] *Outer billiards on Regular Polygons*, [arXiv:1311.6763](https://arxiv.org/abs/1311.6763) are a salient feature of the large scale evolution of the web W . The plots above show these rings superimposed on the black extended edges of N – which form what we call the level-0 web. These rings evolve in the very early web and the plots here are only about level 50. The magenta-blue distinction is used to demonstrate that each inter-ring region is invariant, so no point in a black, blue or magenta region can have an orbit that leaves that region. We will see that this occurs because the D tiles in these rings share common vertices as shown by the blue polygons above. Therefore the rings are τ -invariant and in particular the initial Ring 0 shows that the canonical ‘inner star region’ formed by the truncated edges of N is invariant. We believe that this region can serve as a ‘template’ for global dynamics.

(ii) As a first step we show that each ring is generated by a ‘seed’ D tile on the horizontal axis, and in particular the first ring (which we call Ring 0) is generated by the left-most D tile in the First Family of N . Every such family has both left and right-side D tiles and by reflective symmetry it is sufficient to consider just the left-side D. In Section 4 we show how each $S[k]$ is formed in the early web and this analysis will be repeated below in the context of $N = 8$ to show that as the default extended edges of N are further extended, the D tile edges will generate matching D tiles above and on the left. These horizontal D tiles are what we call the ‘seed’ tiles and for N even their spacing is the same as N to D , but for N odd it is twice this N - D spacing.

(iii) Once the seed D tiles are known, iterating the centers under τ will determine their period and ‘step-sequence’. All the tiles in the orbit of the seed D will share the ‘same’ step-sequence so the centers of the D tiles can be generated by iterating cD under τ^k where k is the sum of the step sequence.

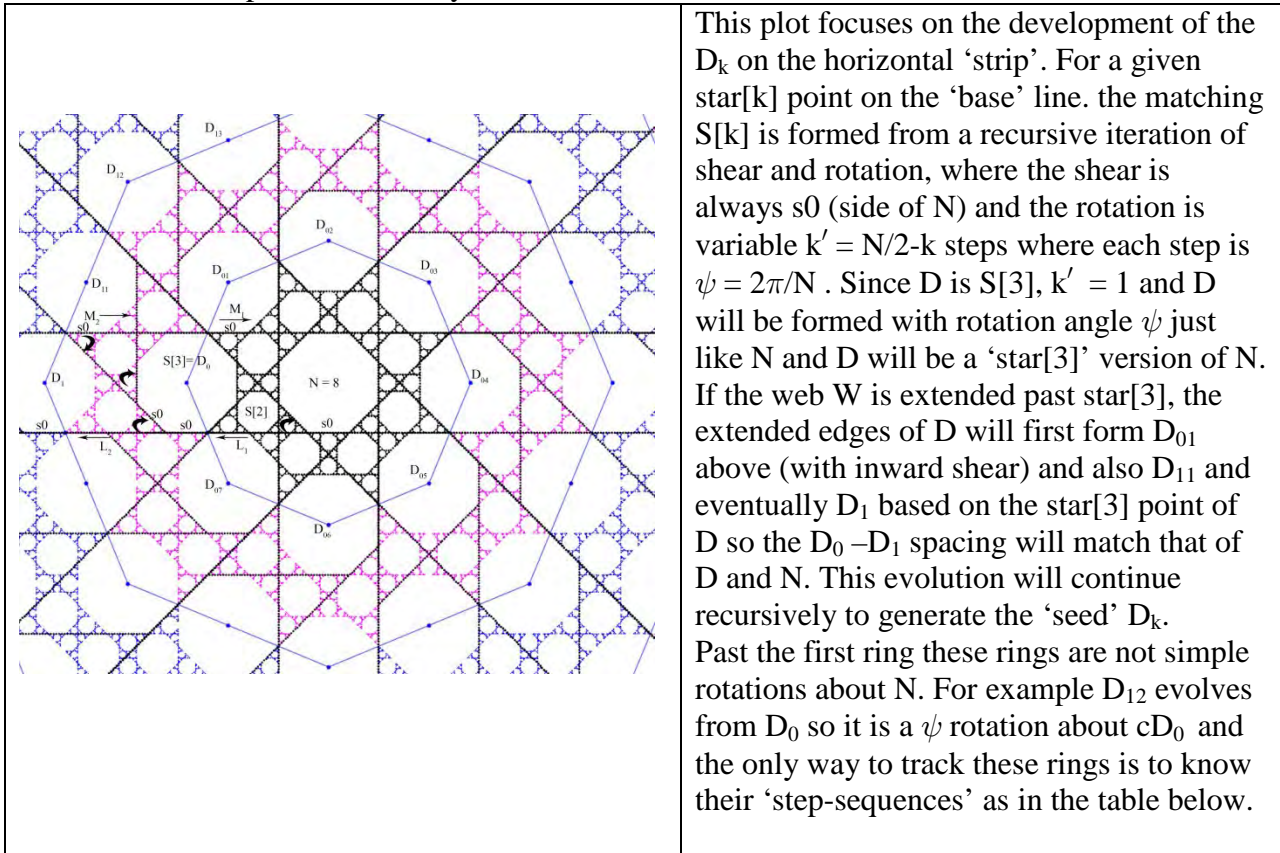
(iv) When N is odd the ring step-sequences are dominated by the step-3 ‘return map’ τ^2 . This occurs because the matching displacement is exactly the radius of D . This allowed us in [H2] to generate a ‘Pinwheel’ map for $N = 7$ to reduce the dynamics to just the horizontal ‘strip’, but the N -even case is quite different and the Pinwheel maps of [S2] are of little value because now the step sequences have the form τ^k for $k > 2$. When N is twice-even the step-sequences are ‘well-behaved’ and Ring k can be generated from the seed D tile by iterations of τ^{k+1} where $k+1$ is the length of the step-sequence of D . But the rings in the twice-odd case will decompose into 2 cycles when the step-sequence length is even, So every even ring will be the (regular) ‘weave’ or ‘Riffle’ of two cycles of equal length. The first cycle is determined by the D seed and the second is based on a ‘neighbor’ which is a copy of D rotated about its center. Here are examples.

N twice-even using N = 8

In this case only we will indicate how the ‘seed’ D tiles form in the web. The other cases are similar because the early web is very predictable. In Section 4 we show how the S[k] First Family tiles evolve in the early web W. This evolution is a form of ‘cobwebbing’ which is commonly used to describe the evolution of a function of one variable by using the $y = x$ line to swap domain and range and continue the iteration. For a rational piecewise isometry like τ this cobwebbing typically involves a translation or ‘shear’ followed by a (variable) rotation to swap domain and range and prepare for the next iteration. Starting with the star[1] point of N, a shear of s_0 (side of N) followed by a rotation $\psi = 2\pi/N$ will generate N, and the S[k] have the same shear but they are based on star[k] and the corresponding rotation angle is $k'\psi$ where $k' = N/2-k$ for N even and $N-2k$ for N odd.

Here for $N = 8$ we are primarily interested in the evolution of the maximal S[3] tile which we call D or D_0 . The matching web rotation angle is $k' = N/2-k = 1$, so D will evolve just like a displaced copy of N and the displacement is star[3] of N – which we call GenStar[N], because the matching scale[3] = $\text{Tan}[\pi/N]/\text{Tan}[3\pi/N]$ is GenScale[N]. This evolution is described below where we indicate how the next D_1 will form from D_0 playing the role of N. This continues recursively to generate endless ‘seed’ D tiles at intervals D to N which is $\text{Tan}[\pi/N] + \text{Cot}[\pi/N]$. When N is odd k' is doubled and the relative ring spacing is $2(\text{Tan}[\pi/N] + \text{Cot}[\pi/2N])$

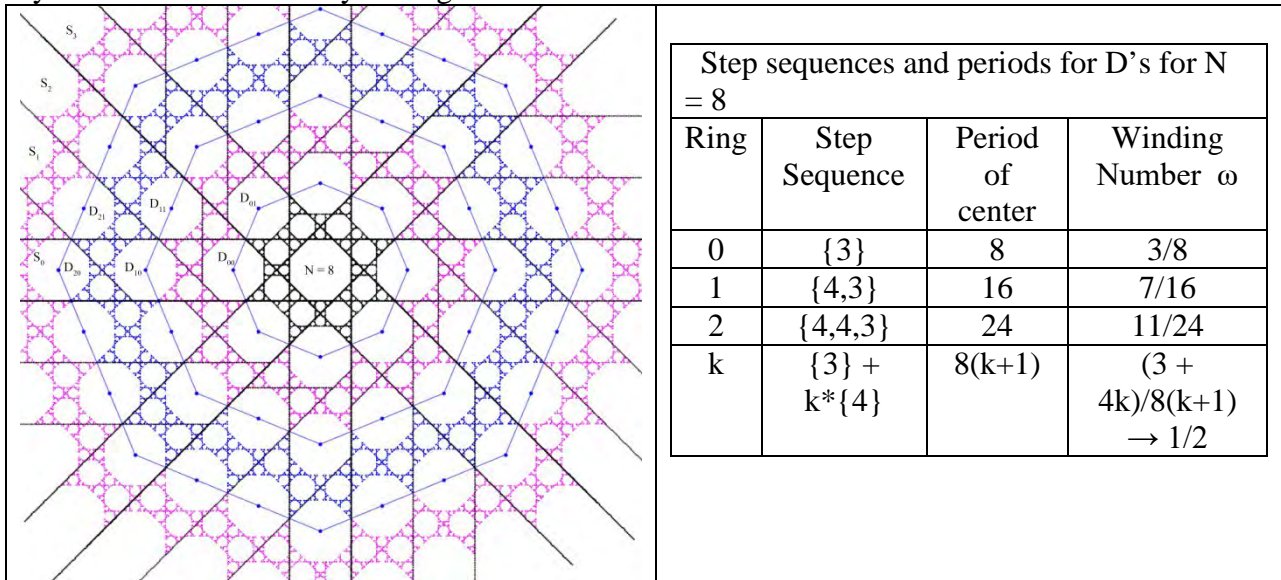
.Below is a description of the early web evolution for $N = 8$



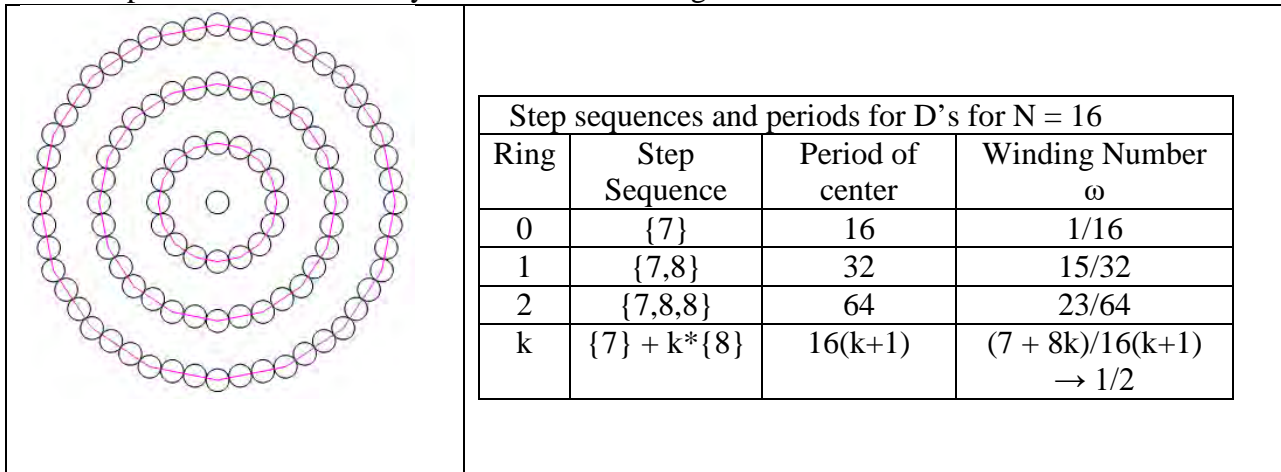
Based on these results we can assume that for any regular N-gon the ‘seed’ D_k are known. We also know that the rings evolve in a recursive fashion so the transition from Ring 0 to Ring 1 should be a template for all rings. For any Ring k it is easy to verify the dynamics by simply iterating cD_k and noting the step sequence and period.

For all N-gons the D tile of Ring 0 has ‘maximal ‘step sequence’ $\{\text{Half}N\}$ which is $N/2-1$ for N even and $\text{Floor}[N/2]$ for N odd . For Ring 1 the new maximal step is the ‘horizon’ value of $\{\text{Half}N+1\}$. Here for $N = 8$, D has step sequence $\{3\}$ and Ring 1 has step sequence $\{4,3\}$.

For example, Ring 2 will have step sequence $\{4,4,3\}$ and period $8*(k+1) = 24$. Since the sum of steps is 11, first generate $24*11$ points in the orbit of the horizontal ‘seed’ D_2 then take these points mod 11 to get the proper ordering: $\text{Orbit} = V[cD_2, 264]$. Then Ring 2 polygon = $\text{Table}[\text{Orbit}[[k]], \{k, 1, 264, 11\}]$; This guarantees that the 24 D tiles in this ring will share a common vertex with their neighbors because they are indeed ‘step-11’ neighbors of each other in the web. This R2 polygon is shown below. When N is even these rings grow in a centrally symmetric fashion so they are regular.



For comparison here is the very similar table and rings for $N = 16$

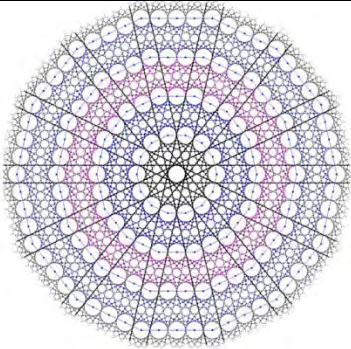
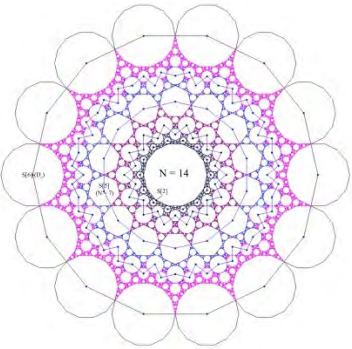
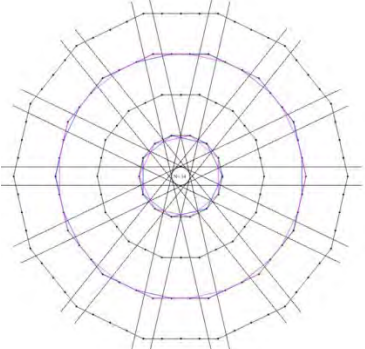


N twice-odd using N = 14

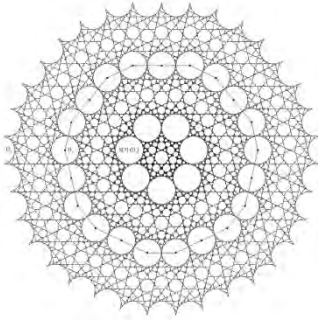
As noted above the web evolution for N twice-odd is less 'forgiving' than N twice-even, so just like the S[3] tile of N = 18, the orbit of D will 'decompose' when its step-sequence has a common factor with N. For example the seed D_2 in Ring 2 has 6+7+7 steps and $\gcd(20,14) = 2$ so this ring will have 2 disjoint cycles of period 21 as shown by the blue and magenta 21-gons. The second cycle will be generated by a rotational 'neighbor' of the seed D tile. This means that Ring 2 will be the 'Riffle' or weave of these two and hence a (regular) 42-gon. Ring 0 with step-sequence {6} is the Riffle of blue and magenta heptagons so it is a regular N-gon. The odd orbits do not decompose so they are easy to generate but it is necessary to know their step sequences to generate the matching 'invariant polygon'. For example to generate Ring 3, over-scan with $K = V[\text{cD3}, 56*27]$; Then Ring 3 polygon = Table[K[[k]], {k, 1, 56*27, 27}]. This guarantees that each D tile in Ring 3 will share a vertex with its step-27 neighbors.

Step sequences and periods for D's for N = 14			
Ring	Step Sequence	Periods of centers	Winding Number - ω
0	{6}	7 & 7	6/14
1	{6,7}	28	13/28
2	{6,7,7}	21 & 21	20/42
3	{6,7,7,7}	56	27/56
k even	{6} + k*{7}	14(k+1)/2 & 14(k+1)/2	$(6 + 7k)/14(k+1) \rightarrow 1/2$
k odd	{6} + k*{7}	14(k+1)	$(6 + 7k)/14(k+1) \rightarrow 1/2$

Invariance can exist for all N-gons at all scales and in [VS] the authors discovered that even some non-regular N-gons will have major rings or secondary rings. When N is twice-odd the embedded N/2 tiles always form secondary 'M' rings inside the D rings as shown below in Ring 0 of N = 14. But these vertex sharing secondary rings do not account for most invariant regions local to N. The majority of these secondary invariant regions local to N are formed from *edge* sharing and not vertex sharing. This issue will be discussed below, but note here that S[3] and S[4] of N and D both share an edge and they define two irregular rings, so this Ring 0 region has 4 secondary rings which affect the local dynamics. For the M-tiles the inter-ring dynamics will be similar to the large scale dynamics of D rings and in both cases reflected points will have dynamics similar to the original. This relates the DS[k] dynamics to the known S[k] case.

The first 4 rings of D tiles	Ring 0 has 4 secondary rings (and N = 23 has 9 such rings)	The major ring polygons
		

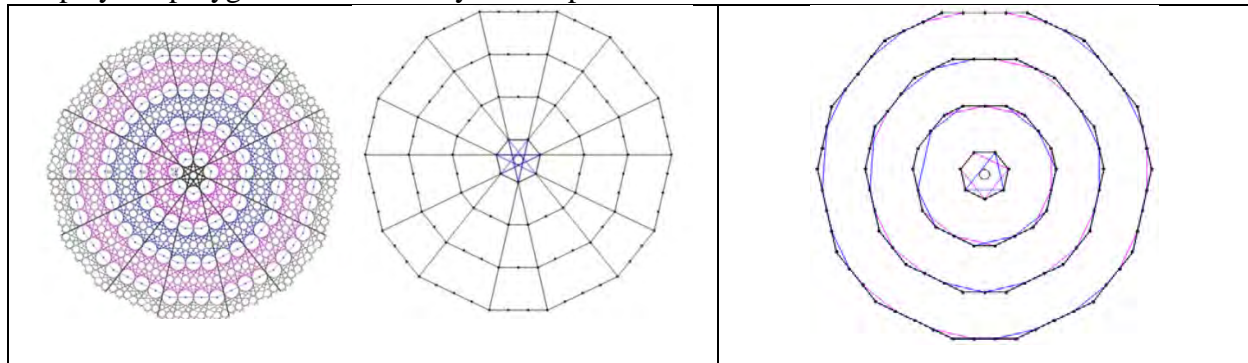
N odd using N = 7

	Step sequences and periods for D's for N = 7			
	Ring	Step Sequence	Period of center	Winding Number ω
	0	{3}	7	3/7
	1	{3,3,4}	21	10/21
	2	{3,3,4,3,4}	35	17/35
k	{3} + k*{3,4}	7(2k+1)	$\frac{3(k+1 + 4k)}{7(2k+1)} \rightarrow 1/2$	

Here HalfN = 3 just like N = 8 so once again D has step sequence {3}, but now the ring growth is a little slower at {3,3,4} for Ring 1. There are no step-4 orbits inside Ring 0 and points that approach D from inside appear to form a Devils Staircase of winding numbers approaching {4}.

When N is odd the rings past Ring 0 are irregular because they are formed from the Riffle (weave) of odd and even cycles of the return map. For Ring 1 the cycle periods are 11 (odd) and 10 (even) so if $K = V[cD1,21]$ is the τ -orbit of $cD1$, then the odd and even cycles are $M1 = \text{Table}[K[[k]], \{k, 1, 21, 2\}]$ and $M2 = \text{Table}[K[[k]], \{k, 2, 21, 2\}]$; Ring 1 polygon = Riffle[$M1$ (blue), $M2$ (magenta)] as shown on the right below. This explains why these invariant polygons are 2N-gons and why they have small winding numbers. Ring 0 is the regular Riffle of a non-regular triangle and an (odd-cycle) quadrilateral. This is reminiscent of the 'projections' from [H3].

Ring 1 can be used as a template for the remaining rings where each side increases by 1 diameter of D, so asymptotically they will be regular 2N-gons. Below are the first 4 rings where for display the polygons are rotated by one step relative to the web.



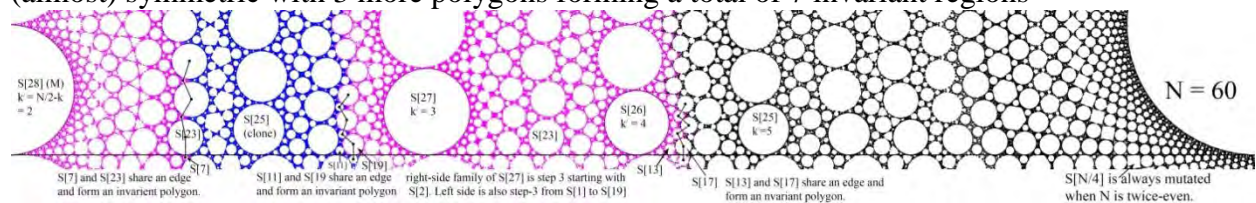
Note: In terms of dynamics, the ring lengths for any regular N-gon clearly have a linear growth rate so it is easy to track the growth of periods by sampling the growth from Ring 0 to Ring 1. Any initial point p_0 in Ring 0 determines an equivalence class $p_k = p_0 + \{k*dx, 0\}$ of points in the horizontal 'strip' where dx is the ring spacing. For N = 7 above, the S[2] tile of D tile has period 21 (which we 'normalize' to 21/7) and the matching p_1 in Ring 1 has (normalized) period 13. This guarantees that p_2 will have period 23 because delta is 10 and the linear growth implies that the period of p_k must be $3 + 10k$. This can always be done on the horizontal strip as long as the periods of p_0 and p_1 are known. However this only applies to the strip and any other point p in Ring k will have to be mapped to a point p_k on the strip. This is theoretically possible but it may be difficult in some cases (like N-odd) where the ring geometry off the strip is complex.

Invariant Regions in the Ring 0 ‘Central Star’ Region

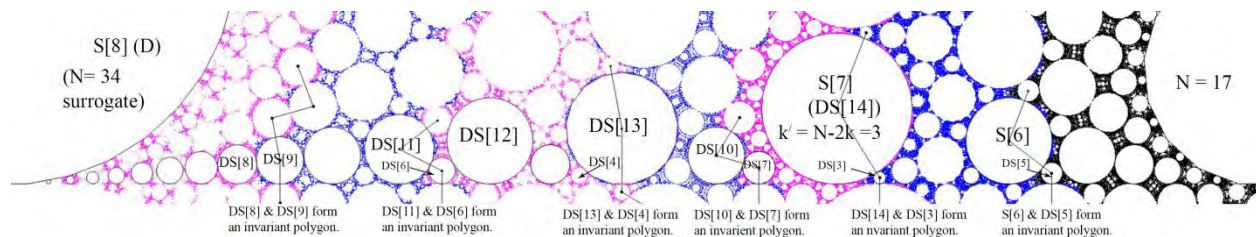
As illustrated by $N = 14$ above, every regular N -gon would be expected to have their own locally invariant regions defined by ‘necklaces’ of tiles. For the rings of D or M tiles we linked the centers to form polygons that ‘thread’ the tiles together and we repeat this here. We loosely refer to these as ‘invariant polygons’, but of course it is the union of the tiles that is invariant. In [SV] Vivaldi referred to the polygons as ‘integral curves’. Unlike the D or M rings, the invariant polygons inside Ring 0 join the centers of adjacent regular tiles that share an *edge*. Among the $S[k]$ family tiles of N , these edge-sharing ‘hugs’ are not as common as vertex sharing, but for these irregular local rings the stronger edge sharing seems to be necessary. The canonical case of edge sharing is N and matching $S[1]$ tile, and all the examples here can be reduced to this case: *If $S[j]$ is any First Family tile of N with web steps k , then if $S[j]$ is ‘promoted’ to N , $S[k]$ (or a clone) has the potential to share an edge with $S[j]$.* So for $N = 17$, the $2N$ -gon $S[6]$ has step sequence $k' = N - 2k = 5$, so a displaced $S[5]$ of D can act as an $S[1]$ and share an edge to form an invariant polygon. For $N = 60$, the $S[17]$ tile of $S[26]$ has $k' = N/2 - k = 13$ so an $S[13]$ can (and will) share an edge as shown below. This web serendipity can be hard to predict and may take a while to occur because the early $S[k]$ will have few secondary $S[k]$ tiles. Often it is easier to work backwards and use random ‘test’ points to generate the approximate bounds of the invariant regions and then look for edge sharing on the fractal boundaries between regions. The Generalized First Family Theorem can be used to find the exact parameters of the tiles involved, but typically the two tiles are from different families and it can be a challenge to reconstruct them. Edge snaring itself may not be a sufficient condition for invariance.

For a case like $N = 17$, these invariance plots can be useful toward understanding how the odd and twice-odd dynamics are related. Here D is a ‘surrogate’ $N = 34$ with predictable $8k+2$ small-scale dynamics, but very little of this structure seems to survive in $N = 17$, even though they share the same web and cyclotomic fields. So the magenta dynamics around D are much better understood than the neighboring blue dynamics beginning at $DS[9]$.

Below are fragments of the 3 right-side invariant polygons for $N = 60$. The left side of M is (almost) symmetric with 3 more polygons forming a total of 7 invariant regions



$N = 17$ below also has 6 invariant polygons and 7 regions. D here acts as a $2N$ -gon ‘parent’ and always has edge sharing with $DS[\text{Floor}[N/2]]$ and right-side neighbor acting as N .



Appendix III

Open Questions

For a regular N-gon the ‘algebraic complexity’ is the rank (dimension) of the vector space determined by the vertices of N. This is also the rank of the maximal real cyclotomic field of N - which is EulerPhi(N)/2 as in the table below.

Algebraic Complexity of regular N-gons for $N \leq 50$

$\phi(N)/2$	1	2	3	4	5	6	8	9	10	11	12	14	15	18	20	21	23
N	3	5	7	15	11	13	17	19	25	23	35	29	31	37	41	43	47
	4	8	9	16	22	21	32	27	33	46	39					49	
	6	10	14	20		26	34	38	44		45						
		12	18	24		28	40		50								
				30		36	48										
						42											

Open Questions: Suppose N is a regular N-gon with limiting web W

(i) As noted in the Introduction the emphasis here has been the early evolution of the web W, but moving forward, the ultimate goal has always been an understanding of the limiting structure of W, and this is primarily a topological issue involving both geometry and dynamics. A salient feature of any symplectic map is the preservation of scales and the potential for a coherent measure theory. Every scale that evolves in W has a matching topology and we would like to know how to generate a broad ‘spectrum’ of these topologies for a given N-gon. In particular where does the classical Housdorff topology sit in this spectrum and does it ‘max-out’ at 2 ? If such a spectrum can be devised, how will the odd and twice-odd cases be related ? See [LV].

(ii) The $8k+2$ Conjecture of [H3] made predictions about the geometry and topology local to the S[1] and S[2] tiles of N. The revised conjecture of [H5] gives formulas for the scaling and periods of D[k] and M[k] tiles in sequences converging to the star[1] point of S[2] and N, so their local topology (and the topology of DS[k] family tiles of S[2]) is known. But except for $N = 10$, we do not know how this is related to the global (multi-fractal) topology of N. For ‘most’ N in this $8k+2$ family it seems that every ‘generation’ of S[2] feels different effects and there is no consensus about the degree of self-similarity across generations. Will this continue or will there be some limiting local two-tiered topology similar to the ‘cubic’ families of $N = 7$ or 9 ?

(iii) Show that if N has algebraic complexity greater than 1 there is at least one ‘exceptional’ point with a non-periodic orbit. (The recent [KRTZ] gives a topological argument for this, which should work for all N, but there is no obvious path toward constructing such points.)

(iv) For any integer $k \geq 0$, the level-k web W_k consists of rays or line segments parallel to the edges of N and in the limit there could be surviving lines or points, such as the ‘star’ points of N. The union of W and its limit points is called the ‘closure’ of W, written W^c . Show that W^c has lebesgue measure 0 just like all the W_k . (This may be very difficult to prove.)

(v) Show that in the large-scale web, the dynamics of every point can be described by an equivalent point inside Ring 0, so this invariant region which we call the ‘central star region’ or ‘generalized star polygon’ is a template for the global web.

References

- [AKT] Adler R.L., B.P. Kitchens, C.P. Tresser, Dynamics of non-ergodic piecewise affine maps of the torus, *Ergodic Theory Dyn. Syst.* 21 (2001), 959–999
- [A] Ashwin P., Elliptical behavior in the sawtooth standard map. *Dynamical Systems*, ISSN 1468-9375, 2001
- [AG] Ashwin P., Goetz A. , Polygonal invariant curves for a planar piecewise isometry. *Transactions of the AMS*, 2006 350:373-390
- [BBW] Baker, A., Birch, B.J., Wirsing, E.A. , On a Problem by Chowla, *Journal of Number Theory*, 5,224-236 (1973)
- [Ca] Calcut J.S. , Rationality of the Tangent Function, preprint (2006)
<http://www.oberlin.edu/faculty/jcalcut/tanpap.pdf>
- [Ch] Chowla, S. The Nonexistence of Nontrivial Linear Relations between Roots of a Certain Irreducible Equation, *Journal of Number Theory* 2, 120-123 (1970)
- [CL] Chua L.O.,Lin T. Chaos in digital filters. *IEEE Transactions on Circuits and Systems* 1988: 35:648-658
- [Co] Coxeter H.S., *Regular Polytopes* (3rd edition - 1973) Dover Edition, ISBN 0-486-61480-8
- [D] Davies A.C. Nonlinear oscillations and chaos from digital filter overflow. *Philosophical Transactions of the Royal Society of London Series A- Mathematical Physical and Engineering Sciences* 1995 353:85-99
- [De] Devaney, R.L., *An Introduction to Chaotic Dynamical Systems*, (2nd edition - 2003) Westview Press, ISBN 0-8133-4085-3
- [G1] Gauss, Carl Friedrich, *Disquisitiones Arithmeticae*, Springer Verlag, Berlin, 1986 (Translated by Arthur A. Clark, revised by William Waterhouse)
- [G2] Gauss, Carl Friedrich; Maser, Hermann (translator into German) (1965), *Untersuchungen über höhere Arithmetik (Disquisitiones Arithmeticae & other papers on number theory) (Second edition)*, New York: Chelsea
- [Gi] Girstmair, K., Some linear relations between values of trigonometric functions at $k\pi/n$, *Acta Arithmetica*, LXXXI.4 (1997)
- [Go] Goetz A. , Dynamics of a piecewise rotation. *Discrete and Cont. Dyn. Sys.* 4 (1998), 593–608. MR1641165 (2000f:37009)

- [GP] Goetz A , Poggiaspalla G., Rotations by $\pi/7$. Nonlinearity 17(5) (2004), 1787–1802.MR2086151 (2005h:37092)
- [GT] Gutkin, E , Tabachnikov,S , Complexity of piecewise convex transformations in two dimensions, with applications to polygonal billiards on surfaces of constant curvature, Mosc. Math. J. 6 (2006), 673-701.
- [Ha] Harper P. G., Proc. Phys.Soc., London (1955) A 68, 874
- [Has] Hasse, H., On a question of S. Chowla, Acta Arithmetica, XVIII (1971)
- [H1] Hughes G.H. Henon mapping with Pascal, BYTE, The Small System Journal, Vol 11, No 13. (Dec.1986) 161-178
- [H2] Hughes G.H., Outer billiards, digital filters and kicked Hamiltonians, [arXiv:1206.5223](https://arxiv.org/abs/1206.5223)
- [H3] Hughes G.H., Outer billiards on Regular Polygons, [arXiv:1311.6763](https://arxiv.org/abs/1311.6763)
- [H4] Hughes G.H., First Families of Regular Polygons [arXiv: 1503.05536](https://arxiv.org/abs/1503.05536)
- H[5] Hughes G.H. First Families of Regular Polygons and their Mutations [arXiv: 1612.09295](https://arxiv.org/abs/1612.09295)
- [H6] Hughes G.H. Edge Geometry of Regular Polygons Part 1 [arXiv: 2103.06800](https://arxiv.org/abs/2103.06800)
- [H7] Hughes G.H. Edge Geometry of Regular Polygons Part 2 [arXiv: 2407.05937](https://arxiv.org/abs/2407.05937)
- [KRTZ] Kanel-Belov A. , Rukhovich P., Timorin V., Zgurskii V. Aperiodic points for dual billiards [arxiv:2311.09643](https://arxiv.org/abs/2311.09643)
- [LKV] Lowenstein J. H., Kouptsov K. L. and Vivaldi F. , Recursive tiling and geometry of piecewise rotations by $\pi/7$, Nonlinearity 17 1–25 MR2039048 (2005f:37182)
- [L] Lowenstein, J.H. Aperiodic orbits of piecewise rational rotations of convex polygons with recursive tiling, Dynamical Systems: An International Journal Volume 22, Issue 1, 2007
- [LV] Lowenstein J.H., Vivaldi F, Approach to a rational rotation number in a piecewise isometric system [/arxiv.org/abs/0909.3458v1](https://arxiv.org/abs/0909.3458v1)
- [Mo1] Moser J.K., On invariant curves of area-preserving mappings of an annulus, Nachr. Akads. Wiss, Gottingen, Math. Phys., K1, (1962)
- [Mo2] Moser J.K., Is the Solar System Stable ? The Mathematical Intelligencer, (1978) Vol. 1, No. 2: 65-71
- [N] Niven I., *Irrational Numbers*, Carus Mathematical Monographs, 11,M.A.A. 1956

[P] Prudnikov, A.P., Brychkov, Y.A., Marichev, O.I., *Integral and Series*, Vol 1, Gordon and Breach, New York, 1986

[Ru] Rukhovich, Philipp Outer billiards outside regular polygons – tame case
Izvestiya: Mathematics 2022, Volume 86, Issue 3, 508–559

[S1] Schwartz R.E., Unbounded orbits for outer billiards, *Journal of Modern Dynamics* 3 (2007)

[S2] Schwartz R.E., Outer billiards, arithmetic graphs, and the octagon. [arXiv:1006.2782](https://arxiv.org/abs/1006.2782)

S[3] *Survey Lecture on Billiards*: ems.press/276/5474

[SM] Siegel C.L, Moser, J.K. *Lectures on Celestial Mechanics*, Springer Verlag, 1971, ISBN 3-540-58656-3

[Sg1] Siegel C.L. *Iteration of (Complex) Analytic Functions*, *Ann of Math* 42: 607-612

[Sg2] Siegel C.L., *Transcendental Numbers*, Princeton University Press, 1949 - 102 pages

[St] Stillwell J., *Numbers and Geometry*, UTM, Springer Verlag, 1991

[T] Tabachnikov S., On the dual billiard problem. *Adv. Math.* 115 (1995), no. 2, 221–249.
MR1354670

[SV] Shaidenko A., Vivaldi F., Global stability of a class of discontinuous dual billiards. *Comm. Math. Phys.* 110, 625–640. MR895220 (89c:58067)

Links

(i) The author's web site at DynamicsOfPolygons.org is devoted to the outer billiards map and related maps from the perspective of a non-professional. The items below are available there.

(ii) A Mathematica notebook called [FirstFamilyFull.nb](#) will generate the First Family and related star polygons for any regular polygon. It is also a full-fledged outer billiards notebook which works for all regular polygons. This notebook includes the Digital Filter map and the Dual Center map. The default height is 1 to make it compatible with the Digital Filter map. This notebook is not necessary to implement the Digital Filter or Dual Center maps, but it may be useful to have a copy of the matching First Family to be used as reference.

(iii) [Outer Billiards notebooks](#) which include the Non-regular case as well as inner-billiards.

(iv) For someone willing to download the free Mathematica [CDF reader](#) there are many 'manipulates' that are available at the Wolfram Demonstrations site - including an [outer billiards](#) manipulate of the author and two other manipulates based on the author's results in [H2].